Topics on Thickness and Ropelength

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Abstract

Roughly speaking, the thickness of a knot is the supremum thickness of a rope in $\mathbb{R}^3$ that realize this particular knot. As expected, the thickness for a knot type is then defined to be the supremum of thickness of representatives in the equivalence class. With such an intrinsic definition, thickness is undoubtedly applicable to several aspects. As a result, there are numerous papers concerning the investigations on ropelength : the reciprocal of thickness.

In the paper "On the Minimum Ropelength of Knots and Links"[5], the very fundamental and essential question of the existence of a ropelength minimizer of a given knot type was solved completely. It was also shown that such a minimizer is $C^{1,1}$ instead of C^{∞}. In addition, some simple minimizers were stated. With the aid of these examples, minimizers are not smooth and unique generally. In practical situations, knots are always represented in polygonal forms for computational purposes. To compute the thickness of a polygonal knot, an Oct-tree based algorithm called "Octrope" with complexity $O(n \log n)$ was introduced in "A Fast Octree-Based Algorithm for Computing Ropelength"[12]. Concerning the problem of ropelength, there were much effort placed on investigating the upper and lower bound on ropelength with respect to crossing number. Two innovative approaches, namely arc-presentation[8] and Hamiltonian knot projection[9], were then designed to give a better estimation for the upper bound.

ACKNOWLEDGMENTS

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Finally, I would like to thank my family for their support throughout my study.

Contents

Chapter 1

Introduction

This thesis mainly consists of five sections : fundamentals of thickness, ropelength minimizer, thickness computation, arc presentation and Hamiltonian knot projection.

In the first section : fundamentals of thickness, the first definition of thickness of a particular knot in $\mathbb{R}^3$ as "injectivity radius" divided by arc-length is introduced. Roughly speaking, injectivity radius is the supremum of radius of its tubular neighborhood such that there is no self intersection. As usual, the thickness of a knot class is defined to be the supremum of all knot representatives inside. Ropelength is then defined to be the reciprocal of thickness. In addition, a theorem relating the injectivity radius and the maximum curvature as well as the minimum distance of double critical points is stated. By the definition of injectivity radius, a polygonal representative of a knot with thickness given can be found easily. Finally, some equivalent definitions for injectivity radius are given as a preparation of the next section.

The major problem concerned in the section : ropelength minimizer is the fundamental problem of ropelength : the existence of ropelength minimizer. At the beginning of the chapter, the crucial concept "secant map" is described.

By the virtue of the other equivalent definitions of injectivity radius, the main lemma stating that "$r(L) > 0 \Leftrightarrow L \in C^{1,1}$" can be proved efficiently. With the aid of some analytical techniques, it can be proved that ropelength minimizer of any tame link type exists. Moreover, some simple ropelength minimizers are given. One of them illustrates that ropelength minimizer can not be assumed to be smooth in general and one illustrates that ropelength minimizer may not be unique.

The third section will be devoted to the algorithm for computing thickness of polygonal knots. For practical reasons, several characteristics of a given knot can only be computed efficiently by polygonal simulation. Therefore, an efficient algorithm for thickness computation will be elaborated in depth. Firstly, as the theorem relating injectivity radius and curvature and double critical points in chapter 2, the polygonal counterpart will be described. Since the minimum radius of curvature can be computed without difficulty, the concentration is placed on the computation of the minimum distance of double critical points. The **Octrope** algorithm given in [12] will be explained, with an example. This algorithm is, naturally, of order $O(n \log n)$. Due to some theoretical considerations, some minor improvements are given at the end of the chapter.

Because of the abstractness of the definition of ropelength, ropelength minimizers, though proved existed, are generally difficult to find. Accordingly, the problem of ropelength bounds was being investigated widely. The last two sections : arc presentation and Hamiltonian knot projection, two different innovative approaches relating ropelength bounds with crossing numbers, will therefore concentrate on this issue. As a matter of fact, arc presentation gives an upper bound of ropelength as $Cross([L])^2$ and Hamiltonian knot projection gives one as $Cross([L])^{\frac{3}{2}}$.

Chapter 2

Fundamentals of thickness

There are three sections in this chapter : definition of thickness, basic theorem and corollaries, equivalent definitions. In the first section, injectivity radius is defined as the supremum of radius of tubular neighborhood such that the exponential map from its normal bundle is injective. The thickness of a particular knot will then be defined as its injectivity radius divided by its arc-length. As expected, the thickness of a knot class is defined to be the supremum of all knot representatives. Finally, the definition of ropelength as reciprocal of thickness will be mentioned as well.

The second section focus on the basic theorem concerning the injectivity radius. Naturally, the injectivity radius will be controlled by two factors, one as local and one as global. The local factor is the local radius of curvature. The global factor is the distance of double critical points : a pair of points (x, y) such that both tangents at x and y are orthogonal to the line segement xy. Obviously, the injectivity radius will be bounded above by these two factors. The basic theorem points out that these factors are the only controlling factors. In addition, for a knot with injectivity radius given, a polygonal representative can be found. A corollary stating that the ropelength, as a knot energy function, is strong will also be given.

The injectivity radius can indeed be defined in two other ways : the infimum of radius of circles defined by three distinct points and $Reach(L)$. The final section is to show that they are all the same. This result will then be used in the next chapter to prove the existence of ropelength minimizer.

Assumptions :

1. All knots and links are tame.
2. All knots and links are in $C^{1,1}$.

Notations :

K	Knot
$[K]$	Knot class of K
$p : \mathbb{R} \to \mathbb{R}^3$	Arc-length parametrization
$L(K)$	Arc-length of K (period of p)
$\mathbb{T}$	p' (unit tangent)
κ	$\|\|\mathbb{T}'\|\|$ (curvature)
E	$\{(p(s), v) \in K \times \mathbb{R}^3 : \quad s \in \mathbb{R} \text{ } and \text{ } \mathbb{T}(s) \cdot v = 0\}$ (total space of the normal bundle)
$\overline{E}_r$	$\{(p, v) \in E : \|\|v\|\| \leq r\}$
$exp : E \to \mathbb{R}^3$	exponential map
$\overline{N(A; r)}$	$\{x \in \mathbb{R}^3 : \quad dist(x, A) \leq r\}$

2.1 Definition of thickness

Definition 2.1.1:

$R(K) \triangleq sup\ \{r > 0 : \quad exp\ is\ injective\ on\ \overline{E}_r\}$,

$\tau(K) \triangleq \frac{R(K)}{L(K)}$,

$\tau([K]) \triangleq sup\ \{\tau(K') : \quad K' \in [K]\ and\ K' \in C^{1,1}\}$ (K can be in C^0 here.),

$\tau(K)$ is called the thickness of K,

$\tau([K])$ is called the thickness of $[K]$

Remark :

By Tubular Neighborhood Theorem[14], $\{r > 0 : \quad exp\ is\ injective\ on\ \overline{E}_r\}$ is non-empty. So, $R(K)$ as well as $\tau(K)$ are well-defined. Obviously, $\tau(K) > 0$ for any K. Since $\forall K \in C^0, \exists K' \in [K], \quad K' \in C^\infty$, $\tau([K])$ is well-defined as long as we restrict K' to be $C^{1,1}$.

$R(K)$ is a natural definition in the sense that $\forall r < R(K), \quad Image(exp|_{\overline{E}_r})$ is the "rope" of K with "thickness" r. As expected, it is not hard to show that $\forall r < R(K)$, $Image(exp|_{\overline{E}_r}) = \overline{N(K;r)}$. Moreover, there is an obvious deformation retract from $\overline{N(K;r)}$ to K.

By dividing $L(K)$, $\tau(K)$ is then scaling-invariant. In the definition of $\tau([K])$, as crossing number and bridge number, $\tau([K])$ is a knot-invariant. But, unlike crossing number, $\tau([K])$ is generally not in $\mathbb{Z}$. Roughly speaking, $\tau([K])$ is much harder to find than other integral knot-invariants as a consequence.

Definition 2.1.2:

$RL(K) \triangleq \frac{1}{\tau(K)}$,

$RL([K]) \triangleq inf\ \{RL(K') : \quad K' \in [K]\ and\ K' \in C^{1,1}\}$ (K can be in C^0 here.),

$RL(K)$ is called the ropelength of K,

$RL([K])$ is called the ropelength of $[K]$

Remark :

For similar reasons, $RL(K)$ and $RL([K])$ are well-defined. As an equivalent definition, $RL([K])$ can be defined as $\frac{1}{\tau([K])}$.

2.2 Basic theorem and corollaries

Definition 2.2.1:

$R_1(K) \triangleq \frac{1}{max\ \kappa(s)}$,

$(x_1, x_2) \in K \times K$ is called DCP (pair of double critical points) if $x_1 \neq x_2$ and $\mathbb{T}(x_1) \cdot (x_1 - x_2) = \mathbb{T}(x_2) \cdot (x_1 - x_2) = 0$,

$R_2(K) \triangleq \frac{1}{2}\ min\ \{||x_1 - x_2|| : \quad (x_1, x_2)\ is\ DCP\}$

Remark :

If $K \in C^{1,1} \setminus C^2$, then $\kappa(s)$ may not be defined at some points. But, $R_1(K)$ can still be defined with $max\ \kappa(s)$ replaced by $sup\ \kappa(s)$.

$R_1(K)$ can be interpreted in two ways. Considering an arc of the knot locally as a circular arc (See figure 1a), by the property of curvature, the radius of this circle will be $\frac{1}{\kappa(s)}$. Accordingly, $R_1(K)$ is just the infimum of all these radii. Since the domain can be restricted as $[0, L(K)]$ (which is compact), infimum can be replaced by minimum. The second interpretation is the following. Similarly, by considering the arc as of a circle, $\frac{1}{\kappa(s)}$ is also the radius of the family of circles which have centers on the arc and touch each other at a single point (See figure 1b). As a result, $R_1(K)$ is an upper bound of $R(K)$ by local behavior.

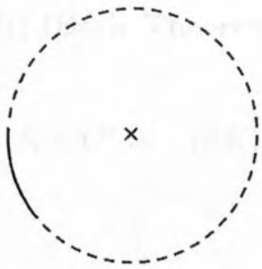


Figure 1a

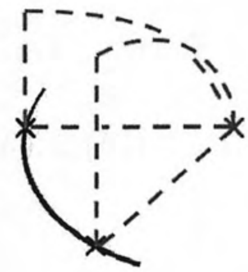


Figure 1b

For $R_2(K)$, DCP is another intrinsic limitation on $R(K)$. Evidently, if (x_1, x_2) is DCP, then $\frac{1}{2}\ ||x_1 - x_2||$ is just the radius of the two circles on the same normal plane of x_1 and x_2 : one at x_1 and one at x_2 and they touch each other at a single point (See figure 2). Unlike $R_1(K)$, $R_2(K)$ is only an upper bound of $R(K)$ by *one of the* global behaviors. Consequently, $R(K) \leq min\ \{R_1(K), R_2(K)\}$.

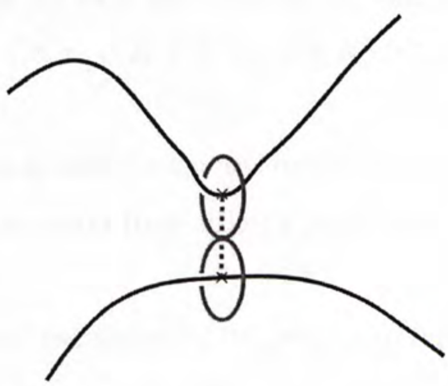


Figure 2

Theorem 2.2.1[1] (Basic Theorem):

$$K \in C^2 \Rightarrow \quad R(K) = min\ \{R_1(K), R_2(K)\}$$

Remark :

By this theorem, if $K \in C^2$, local curvature and DCP are the only two limitations to consider.

By solely the injectivity radius definition, $R(K)$ can hardly be computed. With the aid of this theorem, by using an arc-length parametrization or even a regular parametrization, $R(K)$ can be computed in a reasonably efficient way. Indeed, $R_1(K)$ can be found by finding the maximum of the function $||p''|| : [0, L(K)] \to \mathbb{R}$. For $R_2(K)$, since $\{(x_1, x_2) \in K \times K : (x_1, x_2)\ DCP\}$ is actually the set of (x_1, x_2) attaining the local extremum of the function $||x_1 - x_2|| : (K \times K) \setminus \Delta \to \mathbb{R}$ where $\Delta \triangleq \{(x, x) \in K \times K : \quad x \in K\}$, $R_2(K)$ can be found by considering all the DCP.

In fact, this theorem can also be interpreted in the opposite direction. If $R(K)$ is given, then it must be the result from either a point with maximum $\kappa(s)$ or a DCP.

By following the proof of the theorem, the result can further be extended as this.

$$K \in C^{1,1} \Rightarrow \quad R(K) = min\ \{R_1(K), R_2(K)\}$$

Example 2.2.1:

Considering the trefoil parametrized by

$p : [0, 4\pi] \to \mathbb{R}^3 st.$

$p(t) \triangleq \left(\ (2 - \cos \frac{3}{2}t) \cos t, \ (2 - \cos \frac{3}{2}t) \sin t, \ -\sin \frac{3}{2}t \ \right)$

This is not an arc-length parametrization. So, it is necessary to use the formula $\kappa(t) = \frac{||p'(t) \times p''(t)||}{||p'(t)||3}$ [15].

Here are some numerical results (See figure 3).

$L(K) \approx 31.8986$

$max \ \kappa(t) \approx 0.7$

$R_1(K) \approx 1.42857$

The DCP are $(8.24, 0.14)$, $(9.62, 3.86)$ and $(11.32, 4.51)$.

$R_2(K) \approx 0.831636$

$\therefore R(K) \approx 0.831636$ and $\tau(K) \approx 0.0260712$

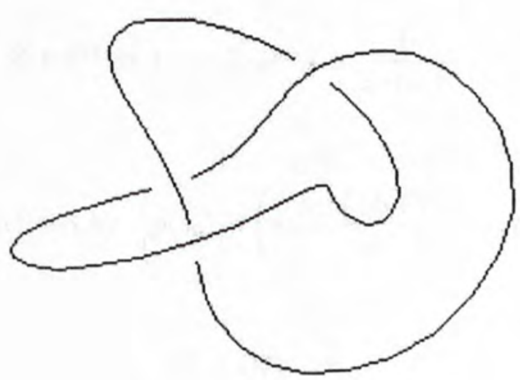


Figure 3

Corollary 2.2.1[1] **:**

$$K \text{ is non-trivial } \Rightarrow \quad \tau([K]) \leq \frac{1}{4\pi}$$

Theorem 2.2.2[1] (Polygonal representative):

$$K \in C^2 \text{ and } n \in \mathbb{Z} \ st. \ n > \frac{1}{\pi\tau(K)}$$

$$\Rightarrow \quad \exists K' \in [K], \quad K' \text{ polygonal knot with } n \text{ segments}$$

Remark :

The lengths of the edges of the polygonal representative K' may be different.

Considering the circle given by $\{(\cos t, \sin t, 0): \quad t \in \mathbb{R}\}$, its thickness is $\frac{1}{2\pi}$. The minimum n in this situation is 3. So, at least for this situation, the n found by this theorem is the best we can get.

Indeed, from the proof in [1], the result can be further refined as this.

$$K \in C^2 \text{ and } n \in \mathbb{Z} \ st. \ n > \frac{1}{\pi\tau(K)}$$

$$\Rightarrow \quad \forall s_0 \in \mathbb{R}, \quad K' \text{ defined by } \left[p(s_0), p\left(s_0 + \frac{L(K)}{n}\right), \cdots, p(s_0 + L(K))\right],$$

$$K' \in [K]$$

Example 2.2.2:

Using the same trefoil in **Example 2.2.1**,

$\tau(K) \approx 0.0260712$

$\therefore n \geq 13$

Choose $t_0 = 0$, the polygonal representative will be

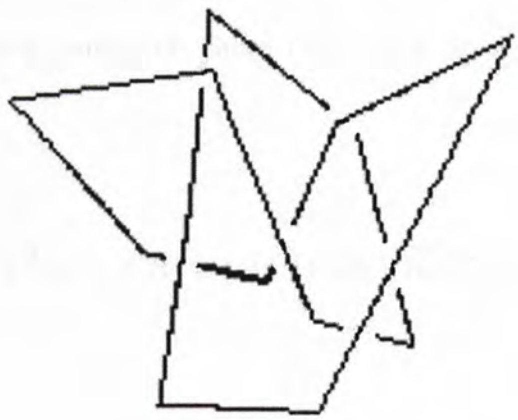


Figure 4

This result is quite unsatisfactory in the sense of finding the minimum number of segments needed for a polygonal representative of the trefoil (The minimum number should be 6).

Corollary 2.2.2[1] :

$$\forall \tau > 0, \exists \text{ finite } [K], \quad \tau([K]) \geq \tau$$

Remark :

While considering RL as a knot energy function[2], this corollary means RL is a strong energy function. ($\forall a > 0, \exists$ finite $[K], \quad RL([K]) \leq a$)

Definition 2.2.2:

$(x_1, x_2) \in (K \times K) \setminus \Delta$ is called SCP (pair of single critical points)
if $\mathbb{T}(x_1) \cdot (x_1 - x_2) = 0$,

$\widetilde{R_2}(K) \triangleq \frac{1}{2}\, min\ \{||x_1 - x_2|| : \quad (x_1, x_2)\ is\ SCP\}$

Remark :

Obviously, SCP is not symmetric. Since $DCP \Rightarrow SCP$, $\widetilde{R_2}(K) \leq R_2(K)$.

Theorem 2.2.3[1] :

$$K \in C^2 \Rightarrow \quad R(K) = min\ \{R_1(K), \widetilde{R_2}(K)\}$$

Remark :

It is a rather surprising result. Another way to interpret the result is this : if $(x_1, x_2)\ is\ SCP$ but not DCP, then (x_1, x_2) will not determine $R(K)$.

For computational purpose, this theorem can not help to improve the algorithm used in the remark of **Theorem 2.2.1 in p.12**.

2.3 Equivalent definitions

Notation :

L Link

Definition 2.3.1:

$x, y, z \in L$ are distinct

$C(x, y, z) \triangleq$ the unique circle defined by x, y, z

$(C(x, y, z) \triangleq$ the line passes through x, y, z if x, y, z are collinear),

$\mathbb{T}_x \triangleq \{x + s\mathbb{T}(x) : \quad s \in \mathbb{R}\}$ (Tangent line at x),

$C(\mathbb{T}_x, y) \triangleq$ the unique circle defined by $\mathbb{T}_x$ and the point y

$(C(\mathbb{T}_x, y) \triangleq \mathbb{T}_x$ if $y \in \mathbb{T}_x)$,

$r(x, y, z) \triangleq$ the radius of $C(x, y, z)$

$(r(x, y, z) \triangleq \infty$ if $C(x, y, z)$ is a line),

$r(\mathbb{T}_x, y) \triangleq$ the radius of $C(\mathbb{T}_x, y)$

$(r(\mathbb{T}_x, y) \triangleq \infty$ if $C(\mathbb{T}_x, y)$ is a line),

$r(L) \triangleq inf\ \{r(x, y, z) : \quad x, y, z \in L\ are\ distinct\}$

Remark :

By the definition of $r(x, y, z)$, $r(L)$ is even defined for $L \in C^0$.

By taking $y \to x$, $\lim_{y \to x} C(x, y, z) = C(\mathbb{T}_x, z)$. If $L \in C^2$, then, by taking $y, z \to x$, it can be locally considered as circular arc. As a result, $\lim_{y,z \to x} r(x, y, z) = \frac{1}{\kappa}$.

Definition 2.3.2:

$X \subseteq \mathbb{R}^3$

$$Reach(X) \triangleq sup\ \{\rho \geq 0 : \quad \forall p \in N(X;\rho), \exists! q \in X, \quad ||p-q|| = dist(X,p)\}$$

Remark :

If $\rho = 0$, then $N(X;\rho) = \emptyset$. So, $Reach(X)$ is well-defined for any X. By Tubular Neighborhood Theorem[14], for any link L, $Reach(L) > 0$. As mentioned, $Reach(L)$ is well-defined even in C^0. But for a polygonal link, the $Reach(L)$ will be determined by the turning points as 0.(See figure 5)

Figure 5

Theorem 2.3.1[5] (Equivalent definitions):

$$r(L) = Reach(L) = \tau(L)$$

Chapter 3

Ropelength minimizer

There are two sections in this chapter : existence of ropelength minimizer, some ropelength minimizers. In the first section, the concept of "secant map" will be given. By this secant map and the equivalent definitions given at the end of chapter 2, the major lemma stating that "$r(L) > 0 \Leftrightarrow L \in C^{1,1}$" can be proved. With some analytical techniques concerning the convergence of sequence of $C^{1,1}$ knots, the ropelength minimizer of any tame knot type is guaranteed to exist.

Ropelength minimizer of a knot type is generally extremely difficult to find. But for some particularly simple link types, ropelength minimizer can be found and is relatively simple. The second section is devoted for this. Consequently, one of these minimizers shows that ropelength minimizer may not be smooth, even C^2 in general. Another minimizer shows that ropelength minimizer may not be unique up to translation, rotation and scaling.

3.1 Existence of ropelength minimizer

Assumptions :

The assumption that all knots and links are in $C^{1,1}$ will be dropped in this section. Instead, all knots and links are assumed to be rectifiable only.

Definition 3.1.1:

$S : (L \times L) \setminus \Delta \to \mathbb{RP}^2$ *st.* $S(x,y) \triangleq \pm \frac{x-y}{||x-y||}$,

S is called secant map

Remark :

The metric on $\mathbb{RP}^2$ is defined as the following.

$l_1 \triangleq \{sp_1 : s \in \mathbb{R}\}$, $l_2 \triangleq \{sp_2 : s \in \mathbb{R}\}$, where $p_1, p_2 \in \mathbb{S}^2$.

$\theta \triangleq$ the angle between p_1 and p_2.

$d(l_1, l_2) \triangleq |\sin\theta|$ (See figure 6).

By taking $|\sin\theta|$, $|\sin\theta_1| = |\sin\theta_2| = |\sin\theta_3| = |\sin\theta_4|$. So, $d(l_1, l_2)$ is well-defined. In addition, it defines a metric on $\mathbb{RP}^2$.

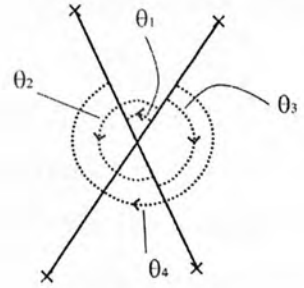


Figure 6

Obviously, $lim_{y \to x} S(x,y)$ exists $\Leftrightarrow \mathbb{T}_x$ exists. In this case, $\mathbb{T}_x = x + lim_{y \to x} S(x,y)$ (differed by a translation).

As a result,

$$L \in C^1 \Leftrightarrow S \text{ can be extended continuously to } L \times L \text{ as } \widetilde{S}$$

and

$$L \in C^{1,1} \Leftrightarrow \widetilde{S} \in C^{0,1}$$

Lemma 3.1.1[5] :

$$r(L) > 0 \Leftrightarrow L \in C^{1,1}$$

Remark :

In the proof in [5], $r(L) > 0$ can as well imply $\widetilde{S}$ is Lipschitz with constant $\frac{1}{2r(L)}$.

By this lemma, L is rectifiable and $r(L) > 0 \Rightarrow L \in C^{1,1} \Rightarrow r(L), Reach(L), \tau(L)$ are well-defined and equal and $\tau(L) = min\ \{R_1(L), R_2(L)\}$ (by **Theorem 2.2.1** in p.12).

Given a link type $[L]$, since $\exists L' \in [L]\ st.\ L' \in C^\infty$ $(\Rightarrow \tau(L) > 0)$,

$sup\ \{\tau(L') : \quad L' \in [L]\ is\ rectifiable\} = sup\ \{\tau(L') : \quad L' \in [L],\ \tau(L') > 0\}$

$= sup\ \{\tau(L') : \quad L' \in [L] \cap C^{1,1}\}$ (by the lemma).

$\therefore$ We can restrict $L \in C^{1,1}$ while finding ropelength minimizer.

Theorem 3.1.1[5] (Existence of ropelength minimizer):

$$\forall L, \exists L' \in [L], \quad RL(L') = RL([L])$$

and

$$\forall L, \forall L' \in [L] \; st. \; RL(L') = RL([L]), \quad L' \in C^{1,1}$$

Remark :

By this theorem, the existence of ropelength minimizer of any tame link type is guaranteed. But, nothing concerning the uniqueness of the minimizer is proved at this stage. In fact, examples showing the non-uniqueness of the minimizer will be given in the next section.

3.2 Some ropelength minimizers

Definition 3.2.1:

$$P_n \triangleq min \{ \quad Length(\gamma): \quad \gamma \text{ is a simple closed curve in } \mathbb{R}^2 \text{ st. } \exists x_1, \cdots x_n \text{ are distinct}, \quad (\forall i \neq j, \quad N(x_i,1) \cap N(x_j,1) = \emptyset) \text{ and } (\forall i, \quad N(x_i,1) \text{ in the region bounded by } \gamma) \quad \}$$

Remark :

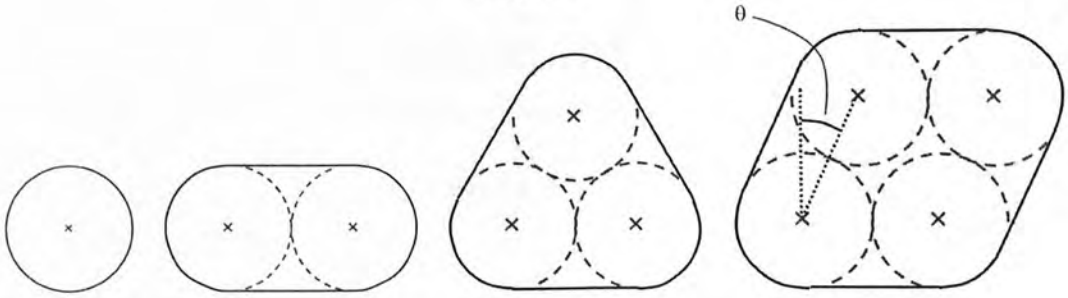
In words, P_n is the length of the shortest curve in $\mathbb{R}^2$ that enclose n unit disks. It is similar to the circle packing problem, but it focuses on the perimeter for finite disks rather than the density on area for infinite disks.

Obviously, there is an equivalent definition.

$$P_n = min \{ \quad Length(\gamma): \quad \gamma \text{ is a simple closed curve in } \mathbb{R}^2 \text{ st. } \exists x_1, \cdots x_n \text{ in the region bounded by } \gamma \text{ and are distinct}, (\forall i \neq j, \quad ||x_i - x_j|| \geq 2) \text{ and } (\forall i, \quad dist(\gamma, x_i) \geq 1) \quad \}$$

For fixed n, the minimizing γ may not be unique. For $n < 3$, γ is unique. But for $n = 4$, there is a one-parameter family of γ. (See figure 7)

Figure 7



$n = 1$	$n = 2$	$n = 3$	$n = 4$
$P_n = 2\pi$	$P_n = 2\pi + 4$	$P_n = 2\pi + 6$	$P_n = 2\pi + 8$
$\gamma \in C^\infty$	$\gamma \in C^{1,1}$	$\gamma \in C^{1,1}$	$\gamma \in C^{1,1}, \ \theta \in [0, \frac{\pi}{6}]$

Theorem 3.2.1[5] (Ropelength lower bound):

$$\tau(L) = 1, \quad L = K \amalg L_1 \amalg \cdots \amalg L_n \ st.$$

$$K \ is \ a \ knot, L_1, \cdots, L_n \ are \ links \ and \ (\forall i, \quad K, L_i \ can \ not \ be \ splitted)$$

$$\Rightarrow \quad L(K) \geq 2\pi + P_n$$

Remark :

Roughly speaking, assuming there is a surface with boundary K, for each i, L_i will puncture this surface at least once. By imagining it as in $\mathbb{R}^2$, K will be a curve surrounding γ that enclose n unit disks. (See figure 8) So, $L(K) \geq 2\pi + P_n$.

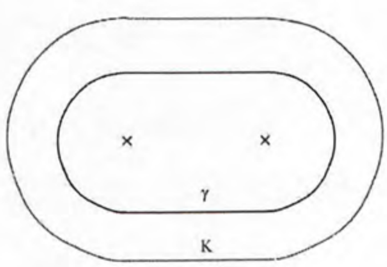


Figure 8

Certainly, there are cases that this bound is the best available. (See figure 9a) But it is also easy to give examples that it is not strict at all. For example, considering a link with a trivial (K) and a trefoil (L) (See figure 9b), since L can not be considered as two links, the theorem gives $L(K) \geq 2\pi + P_1$ instead of $L(K) \geq 2\pi + P_2$.

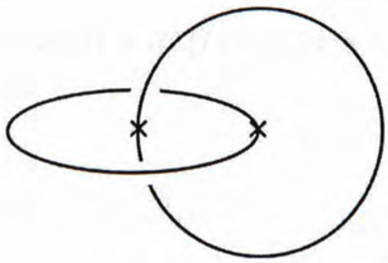


Figure 9a

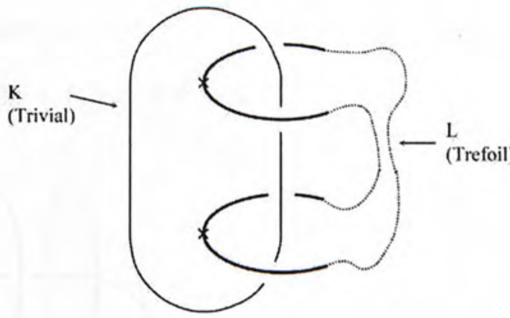


Figure 9b

By this theorem, as long as we keep the link simple enough, the lower bound can be achieved. In this case, it will be a ropelength minimizer.

Example 3.2.1(Ropelength minimizers):

a. Hopf link where the two circles are perpendicular to each other. (See figure 9a)

   $RL([L]) = RL(L) = \frac{2(2\pi+P_1)}{1} = 8\pi.$

b. 2 trivial linked to a trivial. (See figure 10)

   $RL([L]) = RL(L) = 2(2\pi + P_1) + (2\pi + P_2) = 12\pi + 4.$

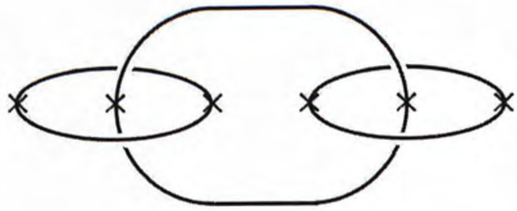


Figure 10

c. 4 trivial linked to a trivial. (See figure 11)

$$RL([L]) = RL(L) = 4(2\pi + P_1) + (2\pi + P_4) = 20\pi + 8.$$

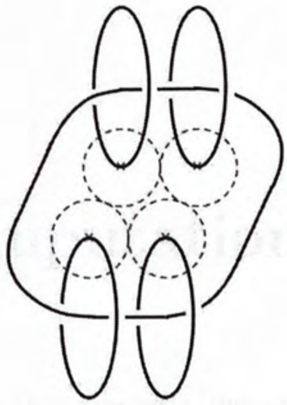


Figure 11

Remark :

As in b, the ropelength minimizer can not be assumed to be C^∞ or even C^2, but it is still possible to be piecewise C^∞.

As in c, the ropelength minimizer may not be unique because the corresponding minimizer γ for P_n may not be unique.

Chapter 4

Thickness computation

There are three sections in this chapter : definitions, **Octrope** algorithm, minor improvements. In the first section, *minRad* is defined as a reasonable counterpart of the minimum radius of curvature in the case of polygonal knots. Similarly, $POCA$ is also defined. Evidently, *minRad* is just of complexity $O(n)$ and so, is not our main concern. A particularly short algorithm (of order $O(n^2)$) for $POCA$ will be given first. To reduce its order, we introduce the concept of *Ramp* and prove a theorem that greatly reduce the number of candidates in checking for $POCA$ against a fixed edge.

In the second section, the three steps of **Octrope** algorithm will be described. The first step : *minRad* computation, as mentioned, is nothing important. The second step : edges partitioning is a procedure to partition the edges by their positions in $\mathbb{R}^3$. Then, in the last step : $POCA$ computation, the condition of whether an edge is inside the *Ramp* of a fixed edge can be checked effectively by the hierarchical structure. An example illustrating the second step will be given at the end of the section.

Throughout the algorithm mentioned, there are two assumptions needed to be considered carefully. The first is to assume the turning angle is between 0 and $\frac{\pi}{2}$. This can be fixed without much difficulty. The second assumption is to identify each edge with its mid-point in the edges partitioning step. Undoubtedly, it is necessary to identify each edge as a single point in order to partition them logically. But, as expected, there are cases that will cause essential problems. So, an improvement will be explained in detail in the last section.

Assumptions :

All knots and links are polygonal.

4.1 Definitions

Definition 4.1.1:

$v : \mathbb{Z} \to \mathbb{R}^3$ *st.* $v(1), \cdots, v(n)$ *are distinct and* $\forall i, \quad v(n+i) = v(i)$

(v exists and is unique if $v(1), \cdots, v(n)$ are given),

$v_i \triangleq v(i)$,

$e_i \triangleq \{(1-t)v_i + tv_{i+1} : \quad t \in [0,1]\}$ *with orientation from* v_i *to* v_{i+1}

(oriented line segment from v_i to v_{i+1}),

$\alpha_i \triangleq \arccos \frac{(v_{i+2}-v_{i+1}) \cdot (v_{i+1}-v_i)}{||v_{i+2}-v_{i+1}|| \ ||v_{i+1}-v_i||}$

(turning angle from e_i to e_{i+1}),

$P_n \triangleq [e_1, \cdots, e_n] \triangleq \bigcup_{i=1}^{n} e_i$ *with orientation induced by* e_i

Remark :

For the algorithm to work properly, an assumption $0 < \alpha_i < \frac{\pi}{2}$ is needed. This assumption is mild and will be discussed with greater depth in the next section.

$P_n = [e_1, \cdots, e_n]$ defines a knot iff $n \geq 3$ and $\forall i$,

$(\forall e_j$ *st.* $e_j \neq e_{i-1}, e_i$ *or* $e_{i+1}, \quad e_i \cap e_j = \emptyset)$ *and* $(e_i \cap e_{i-1} = v_i)$ *and*

$(e_i \cap e_{i+1} = v_{i+1})$

P_n is a knot $\Rightarrow$ P_n has exactly $e_1, \cdots, e_n$ as edges $\Rightarrow$ $[P_n]$ has a polygonal representation with n segments.

Definition 4.1.2:

P_n is a knot,

$$minRad(P_n) \triangleq min \left\{ \frac{||e_i||}{2\tan\frac{\alpha_i}{2}}, \frac{||e_{i+1}||}{2\tan\frac{\alpha_i}{2}} : \quad i \in \mathbb{Z} \right\}$$

Remark :

$minRad(P_n)$ is a reasonable counterpart of R_1. Indeed, $\forall i, \quad 0 < \alpha_i < \frac{\pi}{2} \Rightarrow inf\ r(\mathbb{T}_x, y) = 0$ locally. (See figure 12a) $minRad(P_n)$ is reasonable in the sense of approximating each corner by a circular arc. The arc is taken to be as large as possible in order to maximize the thickness. In this case, for each α_i, the replacing arc is determined by the mid-point of the shorter edge. Accordingly, $minRad(P_n)$ is just the minimum of the radius of all such arcs. (See figure 12b)

Figure 12a

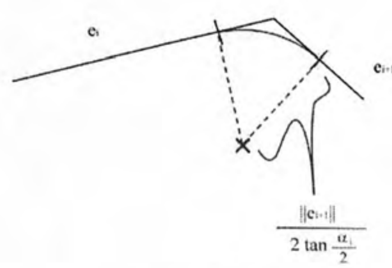


Figure 12b

Definition 4.1.3:

$(x_1, x_2) \in (P_n \times P_n) \setminus \Delta$ *is called POCA* (pair of closest approach)

if (x_1, x_2) *is a local minimum of* $||x - y|| : (P_n \times P_n) \setminus \Delta \to \mathbb{R}$,

$$POCA(P_n) \triangleq \begin{cases} min\ \{||x_1 - x_2|| : \quad (x_1, x_2)\ is\ POCA\} \\ \qquad\qquad\qquad\qquad\qquad if\ \{(x_1, x_2)\ is\ POCA\} \neq \emptyset \\ \infty \\ \qquad\qquad\qquad\qquad\qquad if\ \{(x_1, x_2)\ is\ POCA\} = \emptyset \end{cases}$$

Remark :

According to the remark of **Theorem 2.2.1 in p.12**, this definition of $POCA$ can be thought as an extension given in **Definition 2.2.1 in p.10** for $C^{1,1}$ knots. But there is a minor modification : considering only the local minimum. As a result, $\{(x_1, x_2)\ is\ POCA\}$ may be empty. (See figure 13) In this case, ∞ is assigned to $POCA(P_n)$. In practical situation, $POCA(P_n)$ may be assigned as $2(\frac{||e_i||}{2\tan\frac{\alpha_i}{2}})$ because of the minimality of thickness.

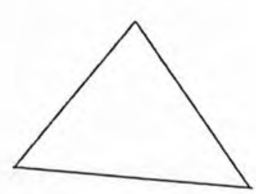


Figure 13

Definition 4.1.4:

$R(P_n) \triangleq min\ \{minRad(P_n),\ \frac{1}{2}\ POCA(P_n)\}$,

$\tau(P_n) \triangleq \frac{R(P_n)}{L(P_n)}$

Remark :

Since $L(P_n)$ is extremely easy to be computed, all algorithms in this chapter concentrates only on $R(P_n)$.

Obviously, the algorithm for finding $minRad(P_n)$ is straightforward and of order $O(n)$. So, to improve the efficiency of finding $R(P_n)$, we concentrate on the algorithm for finding $POCA(P_n)$.

Algorithm 1 for POCA:

$POCA(P_n) := \infty$

For i = 1 to n,

  For j = i+1 to n,

    If $POCA\ (x_1, x_2)\ exists\ for\ e_i, e_j$, then

      If $POCA(P_n) > ||x_1 - x_2||$, then

        $POCA(P_n) := ||x_1 - x_2||$

      End if

    End if

  Next j

Next i

Remark :

Algorithm 1 is intuitive and easy to program. But it is of order $O(n^2)$ which is undesirable. It is not difficult to see that in order to find a $POCA$ with respect to a fixed e_i, only a limited number of edges e_j have to be considered. In this case, the complexity should be considerably reduced.

Definition 4.1.5:

$\forall i, \forall x \in e_i \setminus \{v_i\}, \quad \mathbb{T}^-(x) \triangleq v_{i+1} - v_i,$

$\forall i, \forall x \in e_i \setminus \{v_{i+1}\}, \quad \mathbb{T}^+(x) \triangleq v_{i+1} - v_i$

Remark :

$\mathbb{T}^+(x), \mathbb{T}^-(x)$ are both defined for any $x \in P_n$. In words, $\mathbb{T}^+(x)$ ($\mathbb{T}^-(x)$) is just the tangent of y where $y \to x^+$ (x^-) in the natural sense. (See figure 14)

Evidently, $\forall x \in e_i \setminus \{v_i, v_{i+1}\}, \quad \mathbb{T}^-(x) = \mathbb{T}^+(x)$ and $\forall i, \quad \mathbb{T}^-(v_i) \neq \mathbb{T}^+(v_i)$.

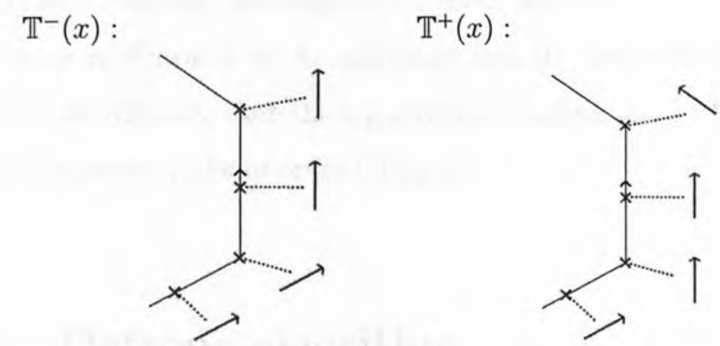


Figure 14

Definition 4.1.6:

$$Ramp_i \triangleq \{x \in R^3 : \quad (x - v_i) \cdot \mathbb{T}^+(v_i) \geq 0 \text{ and } (x - v_{i+1}) \cdot \mathbb{T}^+(v_i) \leq 0\} \cup$$
$$\{x \in R^3 : \quad (x - v_i) \cdot \mathbb{T}^-(v_i) \geq 0 \text{ and } (x - v_i) \cdot \mathbb{T}^+(v_i) \leq 0\}$$

(See figure 15)

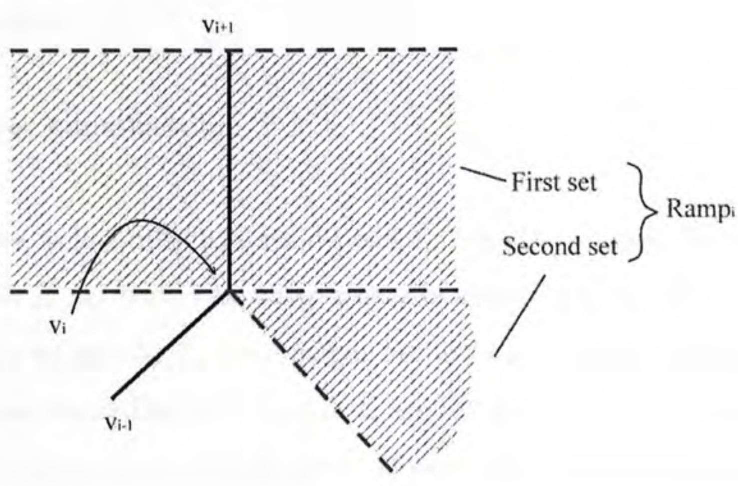


Figure 15

Theorem 4.1.1[12] :

$$(x, y) \text{ is } POCA \text{ of } P_n \text{ and } x \in e_i \setminus \{v_{i+1}\} \Rightarrow \quad y \in Ramp_i$$

Remark :

By this theorem, the edge e_j to check for $POCA$ is limited by $Ramp_i$. If each edge is identified by its mid-point and all these mid-points are partitioned by their coordinates, then the algorithm for finding all e_j which intersects $Ramp_i$ can be expected to be of order $O(log\ n)$.

4.2 Octrope algorithm

Octrope Algorithm

Step 1 :	$minRad$ computation
Step 2 :	Edges partitioning (Octree construction)
Step 3 :	$POCA$ computation

Step 1 : $minRad$ computation

Straightforward.

Step 2 : Octree construction

Theory

Firstly, each e_i is identified by its mid-point $\widetilde{e_i}$. Let B be a box in $\mathbb{R}^3$ containing all $\widetilde{e_i}$. Then, all $\widetilde{e_i}$ are sorted by their x-coordinates to create a list called L_x. Similarly, two more lists L_y, L_z are created. For list L_x, naturally, there will be a value x_0 that the x-coordinates of half of $\widetilde{e_i}$ are less than x_0 and half are above. In the same sense, y_0, z_0 can be obtained. B can thus be divided by (x_0, y_0, z_0) into eight boxes. Inductively, each boxes can further by divided. Let $m \in \mathbb{Z}^+$ be a controlling factor specifying the maximum number of $\widetilde{e_i}$ in a single box.

Consequently, the procedure will terminate in a finite number of steps with each box containing no more than m edges.

Considering an example in $\mathbb{R}^2$.

$\widetilde{e_1}$	$(0,0)$
$\widetilde{e_2}$	$(10,1)$
$\widetilde{e_3}$	$(1,10)$
$\widetilde{e_4}$	$(9,9)$
$\widetilde{e_5}$	$(4,7)$

$m \triangleq 1$

$B \triangleq (0,0)\ to\ (10,10)$

Then, $L_x = (\widetilde{e_1}, \widetilde{e_3}, \widetilde{e_5}, \widetilde{e_4}, \widetilde{e_2})$ and $L_y = (\widetilde{e_1}, \widetilde{e_2}, \widetilde{e_5}, \widetilde{e_4}, \widetilde{e_3})$

In level 1, choose $x_0 \triangleq 3, y_0 \triangleq 3$,

B_{00}	$(0,0)\ to\ (3,3)$
B_{01}	$(0,3)\ to\ (3,10)$
B_{10}	$(3,0)\ to\ (10,3)$
B_{11}	$(3,3)\ to\ (10,10)$

(See figure 16a)

In level 2, only B_{11} is necessary to be divided, choose $x_{0B_{11}} \triangleq 8,\ y_{0B_{11}} \triangleq 8$,

B_{1100}	$(3,3)\ to\ (8,8)$
B_{1101}	$(3,8)\ to\ (8,10)$
B_{1110}	$(8,3)\ to\ (10,8)$
B_{1111}	$(8,8)\ to\ (10,10)$

(See figure 16b)

The results are :

B_{00}	$\{\widetilde{e_1}\}$
B_{01}	$\{\widetilde{e_3}\}$
B_{10}	$\{\widetilde{e_2}\}$
B_{1100}	$\{\widetilde{e_5}\}$
B_{1101}	$\emptyset$
B_{1110}	$\emptyset$
B_{1111}	$\{\widetilde{e_4}\}$

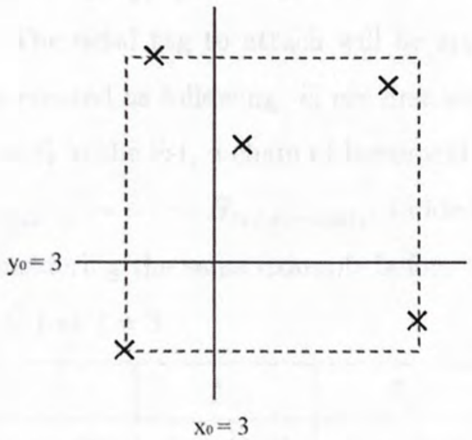


Figure 16a

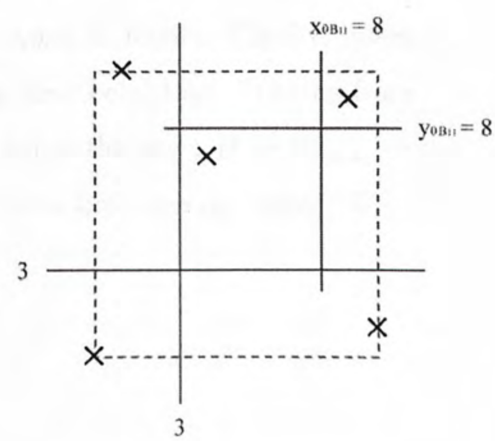


Figure 16b

Remark :

This partitioning is absolutely an intuitive one. Obviously, $\widetilde{e_i}$ are grouped by their positions in $\mathbb{R}^3$. If $Ramp_i$ does not intersect a certain box, then we can simply ignore all $\widetilde{e_i}$ inside. There is a fatal assumption in this procedure : identifying e_i with its mid-point. This problem will be further discussed in the next section.

Implementation

In practical situation, the actual algorithm looks rather different. But the underlying concept is essentially the same.

First of all, L_x, L_y, L_z are created as before. Then, in L_x, m (a controlling factor) consecutive $\widetilde{e_i}$ will be grouped together. Let $l \in \mathbb{Z}^+$ be the unique integer st. $2^{l-1} < number\ of\ groups\ < 2^l$. The groups are then labeled as group 0, group 1, $\cdots$ etc. L_y, L_z are processed similarly. For each $\widetilde{e_i}$, an octal tag will be attached by the following procedure.

A fixed $\widetilde{e_i}$ will belong to a *group* n_x in L_x, *group* n_y in L_y and *group* n_z in L_z. Then, n_x, n_y, n_z are expressed in binary as $x_1x_2\cdots x_l$, $y_1y_2\cdots y_l$, $z_1z_2\cdots z_l$ resp.. The octal tag to attach will be $z_1y_1x_1\cdots z_ly_lx_l$ in binary. Finally, boxes will be created as following. $\widetilde{e_i}$ are first sorted by their octal tags. Starting from the first $\widetilde{e_i}$ in the list, a chain of boxes will be added to the tree : $B \to B_{z_1y_1x_1} \to B_{z_1y_1x_1z_2y_2x_2} \to \cdots \to B_{z_1y_1x_1\cdots z_ly_lx_l}$. (added only if the box does not exist)

Considering the same example before.

$m \triangleq 1 \Rightarrow l = 3$

	L_x	L_y
Group 000	$\widetilde{e_1} = (0,0)$	$\widetilde{e_1} = (0,0)$
Group 001	$\widetilde{e_3} = (1,10)$	$\widetilde{e_2} = (10,1)$
Group 010	$\widetilde{e_5} = (4,7)$	$\widetilde{e_5} = (4,7)$
Group 011	$\widetilde{e_4} = (9,9)$	$\widetilde{e_4} = (9,9)$
Group 100	$\widetilde{e_2} = (10,1)$	$\widetilde{e_3} = (1,10)$

	Octal tag	$(y_1x_1y_2x_2y_3x_3)$
$\widetilde{e_1}$	$000000_{(2)}$	$000_{(4)}$
$\widetilde{e_2}$	$010010_{(2)}$	$102_{(4)}$
$\widetilde{e_3}$	$100001_{(2)}$	$201_{(4)}$
$\widetilde{e_4}$	$001111_{(2)}$	$033_{(4)}$
$\widetilde{e_5}$	$001100_{(2)}$	$030_{(4)}$

Sorted list
$\widetilde{e_1} : 000_{(4)}$
$\widetilde{e_5} : 030_{(4)}$
$\widetilde{e_4} : 033_{(4)}$
$\widetilde{e_3} : 102_{(4)}$
$\widetilde{e_2} : 201_{(4)}$

(Fig. 17a)

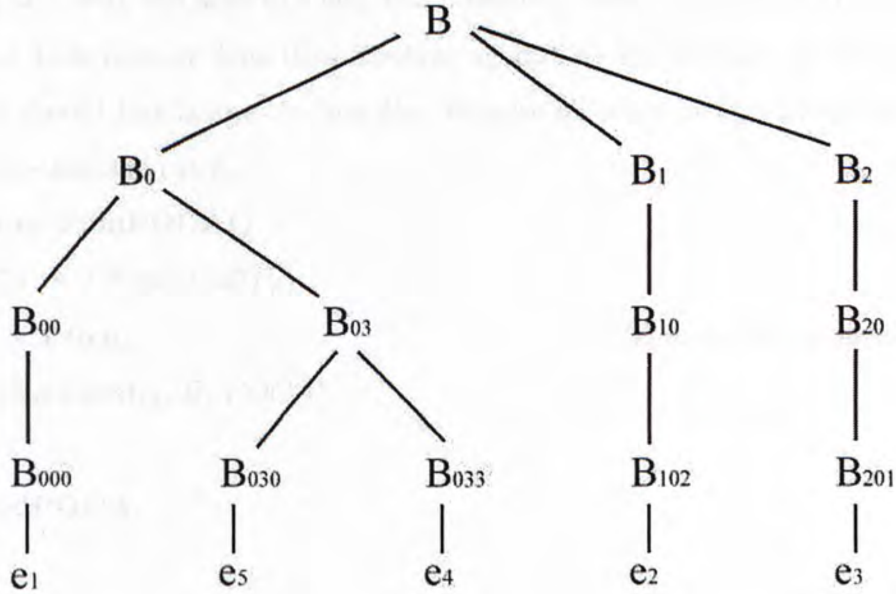


Figure 17b

Remark :

Actually, some clean up processes can be carried out after the creation of the tree. Apparently, the number of boxes and number of levels l can be reduced as long as the criterion imposed by m is satisfied. Indeed, a number of approaches can be applied.

With or without the clean up procedure, there are some properties worth mentioning. Firstly, each $\widetilde{e_i}$ will belong to one and only one box (in the lowest level). Secondly, due to the controlling factor m, each box (in the lowest level) contains at most $m\ \widetilde{e_i}$. Thirdly, no box will be empty. Consequently, the number of boxes in the lowest level is between $\frac{n}{m}$ and n.

Step 3 : *POCA* computation

Since "(x, y) is *POCA*" is symmetric, it is undesirable to check e_i against e_j and then e_j against e_i. This is the reason for using octal tags. By comparing the octal tag of $\widetilde{e_i}$ with the label of a box, it is possible to judge whether all the edges in the box have already done their checking against e_i. For example, $\widetilde{e_i}$ with tag $001010_{(2)}$ should just ignore the box B_{000} because all edges in B_{000} should have already checked against $\widetilde{e_i}$.

Procedure FindPOCA()

$POCA := 2 * minRad(P_n)$

For i = 1 to n, ($\widetilde{e_i}$ sorted by octal tags)

CheckBox(e_i, B, $POCA$)

Next i

End FindPOCA

Procedure CheckBox(e_i, $\widetilde{B}$, $POCA$)

If octal tag of e_i truncated to the level of $\widetilde{B} \leq$ label of $\widetilde{B}$, then

If $N(\widetilde{e_i}; POCA) \cap \widetilde{B} \neq \emptyset$, then

If $Ramp_i \cap \widetilde{B} \neq \emptyset$, then

If $\widetilde{B}$ is of lowest level, then

For each $e_j \in \widetilde{B}$

Check e_i, e_j and update $POCA$ if necessary

Next e_j

Else

For each child box $\widetilde{\widetilde{B}}$ of $\widetilde{B}$

CheckBox(e_i, $\widetilde{\widetilde{B}}$, $POCA$)

Next $\widetilde{\widetilde{B}}$

End if

End if

End if

End if

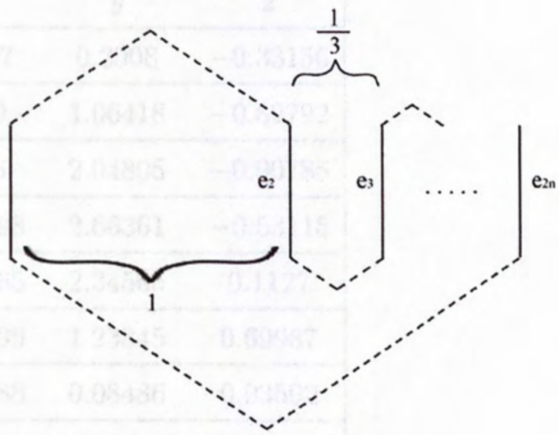
End CheckBox

Remark :

As mentioned, $minRad(P_n)$ is particularly simple and is definitely of order $O(n)$. In step 2, it consists of some sortings ($O(n \ \log n)$), groupings ($O(n)$), labeling ($O(n)$) and a tree creation ($O(n)$). So, it is of order $O(n \ \log n)$.

In step 3, the situation is much more complicated. In natural circumstances, each $Ramp_i$ will only intersects a limited number of boxes that gives the whole procedure the order $O(n \ \log n)$. But for the worst case, since approximately all $Ramp_i$ intersect all other e_j, for each e_i, the whole tree will be gone through once. The order hence raise to $O(n^2 \ \log n)$. (See figure 18)

$n \in \mathbb{Z}^+$



All dotted edges have $\alpha < \frac{\pi}{2}$

Figure 18

Example 4.2.2:

Considering the trefoil in **Example 2.2.1**

$n = 13$

Since some $\alpha_i > \frac{\pi}{2}$, $n \triangleq 26$ is used instead.

In this case, $\forall i, \quad 21.95° < \alpha_i < 70.04°$

$m \triangleq 2 \Rightarrow l = 4$

The points are :

$\widetilde{e}$	x	y	z
1	1.05407	0.2908	−0.33156
2	1.0879	1.06418	−0.82792
3	0.6886	2.04805	−0.90785
4	−0.37198	2.66361	−0.53115
5	−1.60665	2.34565	0.1127
6	−2.22299	1.23845	0.69987
7	−1.94188	0.08486	0.93502
8	−1.21592	−0.56642	0.69987
9	−0.59958	−0.85063	0.1127
10	−0.09615	−1.19184	−0.53115
11	0.6886	−1.58333	−0.90785
12	1.81914	−1.51123	−0.82792
13	2.71684	−0.63865	−0.33156
14	2.71684	0.63865	0.33156
15	1.81914	1.51123	0.82792
16	0.6886	1.58333	0.90785
17	−0.09615	1.19184	0.53115
18	−0.59958	0.85063	−0.1127
19	−1.21592	0.56642	−0.69987
20	−1.94188	−0.08486	−0.93502
21	−2.22299	−1.23845	−0.69987
22	−1.60665	−2.34565	−0.1127
23	−0.37198	−2.66361	0.53115
24	0.6886	−2.04805	0.90785
25	1.0879	−1.06418	0.82792
26	1.05407	−0.2908	0.33156

The results are :

	L_x	L_y	L_z
Group 0	21	23	20
	6	22	3
Group 1	7	24	11
	20	11	12
Group 2	22	12	2
	5	21	21
Group 3	8	10	19
	19	25	4
Group 4	9	9	10
	18	13	13
Group 5	23	8	1
	4	26	22
Group 6	10	20	18
	17	7	5
Group 7	24	1	9
	11	19	14
Group 8	3	14	26
	16	18	23
Group 9	1	2	17
	26	17	6
Group 10	25	6	8
	2	15	15
Group 11	12	16	25
	15	3	24
Group 12	13	5	16
	14	4	7

$\widetilde{e}$	n_x	n_y	n_z	Octal tag
21	0	2	2	0060_8
11	7	1	1	0117_8
20	1	6	0	0221_8
19	3	7	3	0277_8
22	2	0	5	0414_8
10	6	3	4	0532_8
9	4	4	7	0744_8
12	11	2	1	1035_8
1	9	7	5	1627_8
13	12	4	4	1700_8
4	5	12	3	2345_8
18	4	8	6	2540_8
5	2	12	6	2650_8
3	8	11	0	3022_8
2	10	9	2	3052_8
14	12	8	7	3544_8
23	5	0	8	4101_8
24	7	1	11	4157_8
8	3	5	10	4253_8
7	1	6	12	4621_8
25	10	3	11	5076_8
26	9	5	8	5203_8
6	0	10	9	6024_8
17	6	9	9	6116_8
15	11	10	10	7071_8
16	8	11	12	7422_8

Figure 19a :

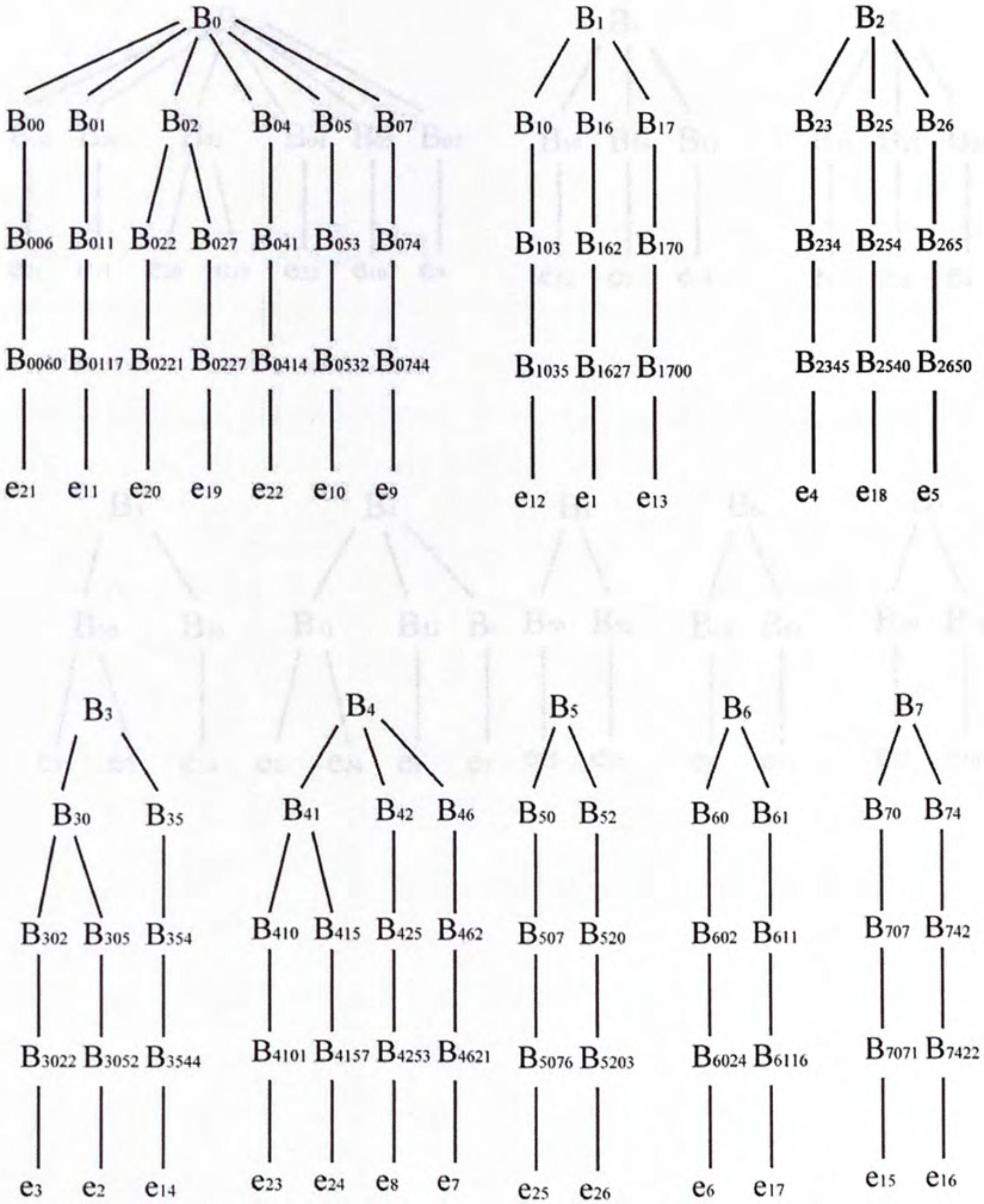
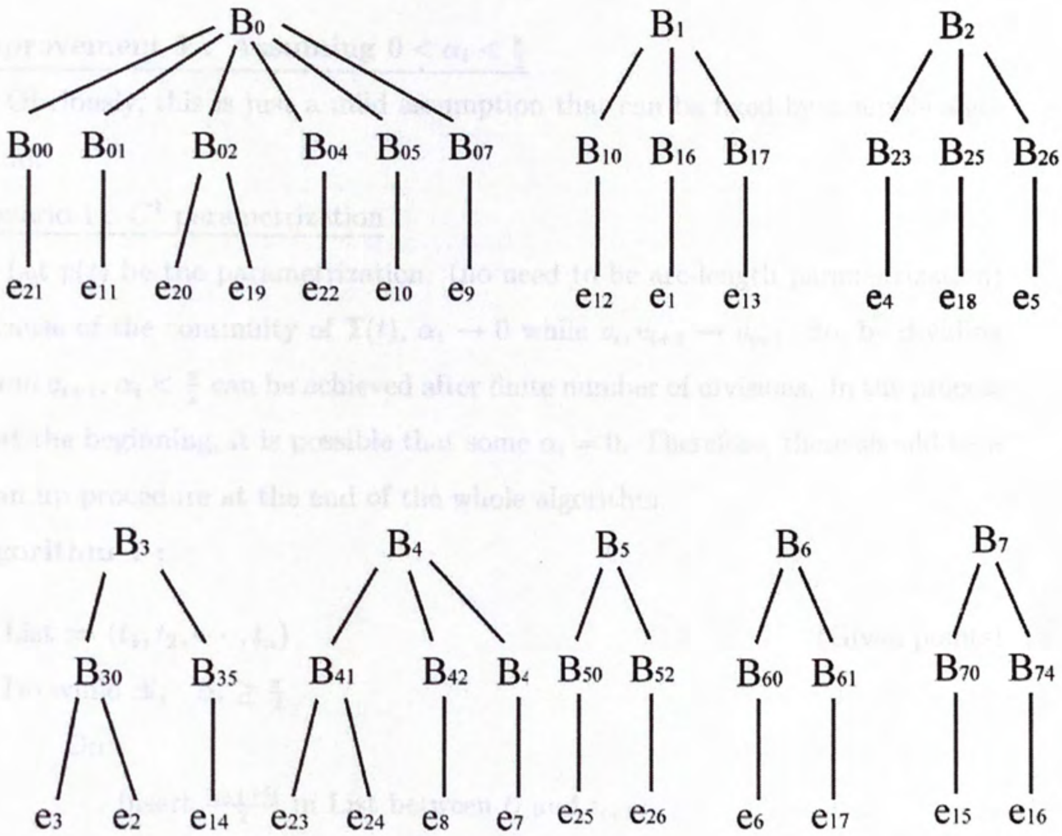


Figure 19b (After clean up):



4.3 Minor improvements

Improvement 1 : Assuming $0 < \alpha_i < \frac{\pi}{2}$

Obviously, this is just a mild assumption that can be fixed by a simple algorithm.

Scenario 1 : C^1 parametrization

Let $p(t)$ be the parametrization. (no need to be arc-length parametrization) Because of the continuity of $\mathbb{T}(t)$, $\alpha_i \to 0$ while $v_i, v_{i+2} \to v_{i+1}$. So, by dividing e_i *and* e_{i+1}, $\alpha_i < \frac{\pi}{2}$ can be achieved after finite number of divisions. In the process or at the beginning, it is possible that some $\alpha_i = 0$. Therefore, there should be a clean up procedure at the end of the whole algorithm.

Algorithm 1 :

List $:= (t_1, t_2, \cdots, t_n)$ (Given points)

Do while $\exists i, \quad \alpha_i \geq \frac{\pi}{2}$

Do

Insert $\frac{t_{i+1}+t_i}{2}$ in List between t_i and t_{i+1}

Insert $\frac{t_{i+2}+t_{i+1}}{2}$ in List between t_{i+1} and t_{i+2}

Loop while $\alpha_i \geq \frac{\pi}{2}$ (calculated according to the new list)

Loop

CleanUp

Remark :

Due to the continuity of $\mathbb{T}(t)$, $\exists d, \forall t_1 < t_2 < t_3$ *st.* $|t_1 - t_3| < d, \quad \alpha_i$ defined by $p(t_1), p(t_2)$ *and* $p(t_3) < \frac{\pi}{2}$. Accordingly, the algorithm will terminate after finite number of iterations.

Another way to deal with this problem is to find d in advance and then dividing the interval of domain of p into intervals with size less than d. $(t_1, \cdots, t_n)$ will then automatically satisfy the criterion that $\alpha_i < \frac{\pi}{2}$ as a result.

The second method will be extremely efficient if d can be found easily. But in the real situation, it may be rather difficult to find such d.

Scenario 2 : Polygonal knot

Let P_n defined by $v_1, v_2, \cdots, v_n$. To fix the problem, approximation is needed. Indeed, this approximation will solve the problem in one step.

First of all, let d be a relatively small positive number (For simplicity, $d \leq min\ \{\frac{1}{3}\ ||v_{i+1} - v_i|| : \quad 1 \leq i \leq n\}$) If $\alpha_i \geq \frac{\pi}{2}$, then v_i is replaced by two points : v_i^- from e_i, v_i^+ from e_{i+1} st. $||v_i^- - v_i|| = ||v_i^+ - v_i|| = d$. As a result, the new turning angles for v_i^- and v_i^+ will both less than $\frac{\pi}{2}$. (See figure 20)

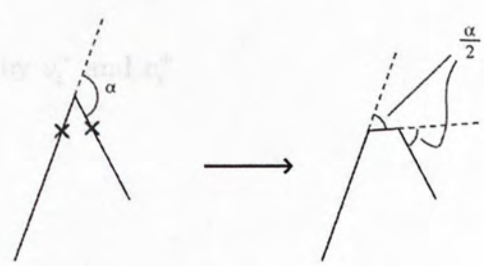


Figure 20

Algorithm 2 :

List $:= (v_1, v_2, \cdots, v_n)$

$d := \frac{1}{3}\ ||v_2 - v_1||$

For i = 2 to n

    If $\frac{1}{3}\ ||v_{i+1} - v_i|| < d$, then

        $d := \frac{1}{3}\ ||v_{i+1} - v_i||$

    End if

Next i

For i = 1 to n

    If $\alpha_i \geq \frac{\pi}{2}$, then

        Replace v_i by v_i^- and v_i^+

    End if

Next i

CleanUp

Remark :

The algorithm is undoubtedly a simple and efficient one with order $O(n)$.

In scenario 1, the more divisions are made, the higher the precision of the simulating polygonal knot will be. In contrast, in scenario 2, the more divisions are made, the lower the precision will be. But as long as d is small enough, the precision will be acceptable.

At this point, it is important to note that $minRad(P_n) \triangleq min\ \{\frac{||e_i||}{2\tan\frac{\alpha_i}{2}}, \frac{||e_{i+1}||}{2\tan\frac{\alpha_i}{2}}\}$ and $R(P_n) \triangleq min\ \{minRad(P_n),\ \frac{1}{2}POCA(P_n)\}$. So, if $\exists i, \quad \alpha_i \geq \frac{\pi}{2}$ and d is taken to be too small, $R(P_n)$ will be very small and is meaningless. More or less, "$\frac{1}{3}$" is a balanced and reasonable choice.

Improvement 2 : Identifying e_i with its mid-point

Assuming P_n is a polygonal knot where $R(P_n)$ is achieved by $\frac{1}{2}$ $POCA(P_n)$, in addition, (x, y) is the only $POCA$ achieving $POCA(P_n)$. Theoretically, $x \in e_i \setminus \{v_{i+1}\} \Rightarrow y \in Ramp_i$. Let $y \in e_j$. By checking e_j against e_i, this $POCA$ can be found. Unfortunately, as e_j is identified as its mid-point $\widetilde{e_j}$, $\widetilde{e_j}$ is than not guaranteed to be in $Ramp_i$. (See figure 21)

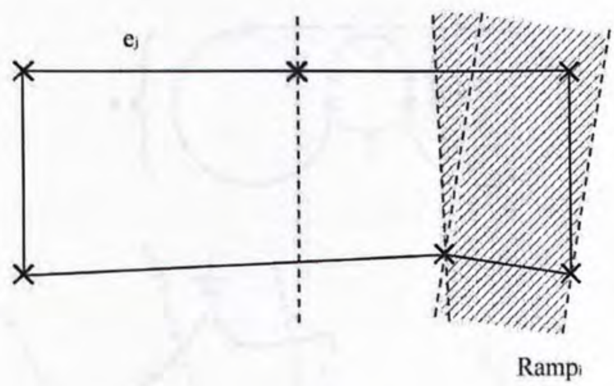


Figure 21

Following the actual algorithm, it is important to note that it is the box to check against $Ramp_i$ instead of $\widetilde{e_j}$. So, if the box $\widetilde{B}$ with lowest level containing $\widetilde{e_j}$ actually intersects $Ramp_i$, this problem can be avoided. But unfortunately, no matter what m is, it is still possible to have $\widetilde{e_j}$ as the right bound of $\widetilde{B}$ and $\widetilde{B} \cap Ramp_i = \emptyset$ as a result.

<u>Approach 1 : By further division</u>

The first approach is to further divide the knot. By further dividing the knot into smaller pieces, hopefully, the critical situation will vanish.

There are a number of problems for this approach. Firstly, even it really work, since the whole algorithm (except Step 1) has to be repeated for each division, the complexity $O(n \log n)$ may increase to an unacceptable level. Secondly, there is no obvious way to determine whether such critical configuration exists or not and hence no control on when to stop. Thirdly, the most fatal problem is : there is a configuration that no matter how short each line segment is, this $POCA$ will still be ignored.

Example 4.3.1:

Considering a $C^{1,1}$ knot K in $\mathbb{R}^2$. (See figure 22)

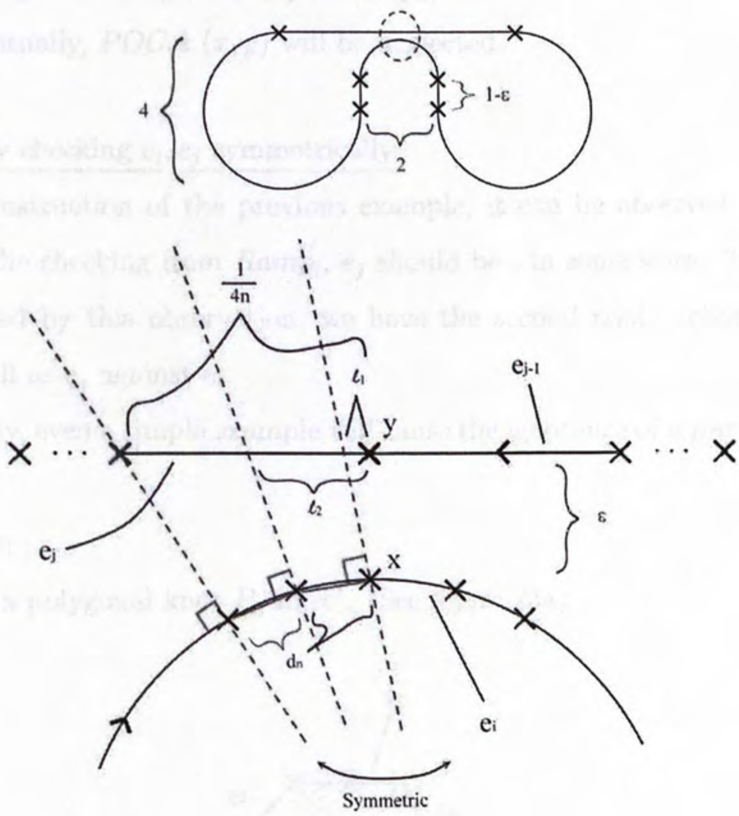


Figure 22

$R(K) = R_2(K)$ and there is one and only one $POCA$ (x, y) determining $R_2(K)$.

$\forall n, \quad d_n \triangleq min\,\{\frac{1}{2}\arctan\frac{1}{4n(1+\epsilon)}, \frac{1}{n}\}$

$l_1 = \epsilon \tan\frac{d_n}{2}$

$l_2 = [\epsilon + 2(1+\epsilon)\cos d_n]\sec\frac{3d_n}{2}\sin\frac{d_n}{2}$

$\Rightarrow 0 < l_1 < l_2 < \frac{1}{4n}$

In this configuration, assuming the octal tag of $e_i <$ octal tag of e_j, then e_j will be checked by $Ramp_i$, but not the reverse. $0 < l_1 < l_2 < \frac{1}{4n} \Rightarrow \widetilde{e_j}, \widetilde{e_{j-1}} \notin Ramp_{i-1}$ *or* $Ramp_i$. Although $\exists k,\ \widetilde{e_j} \in Ramp_k$, no $POCA$ will be given by e_j against e_k. Eventually, $POCA\ (x,y)$ will be neglected.

Approach 2 : By checking e_i, e_j symmetrically

From the construction of the previous example, it can be observed that in order to avoid the checking from $Ramp_i$, e_j should be , in some sense, "longer" than e_i. Inspired by this observation, we have the second trial : checking e_i against e_j as well as e_j against e_i.

Unfortunately, even a simple example will cause the ignorance of a **particular** *POCA*.

Example 4.3.2:

Considering a polygonal knot P_n in $\mathbb{R}^2$. (See figure 23a)

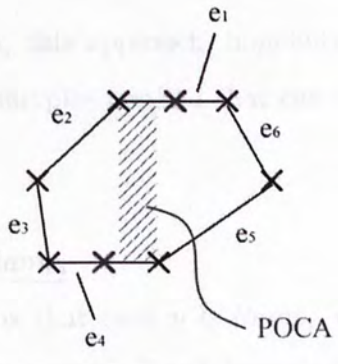


Figure 23a

The only $POCA$ occurs in the shaded region as open intervals on both sides. It is apparent that both e_1, e_4 ignore each other and thus cause the problem.

But it should be noted that $R(P_n)$ in this example is given by $minRad(P_n)$ instead of $POCA(P_n)$. In general, if $R(P_n)$ is given by $POCA(P_n)$, then at a particular $POCA$ (x, y) achieving $POCA(P_n)$, the adjacent edges of e_i and e_j will be "moving away". This will naturally force at least one of the ramp of an adjacent edge to include the opposite edge. (See figure 23b)

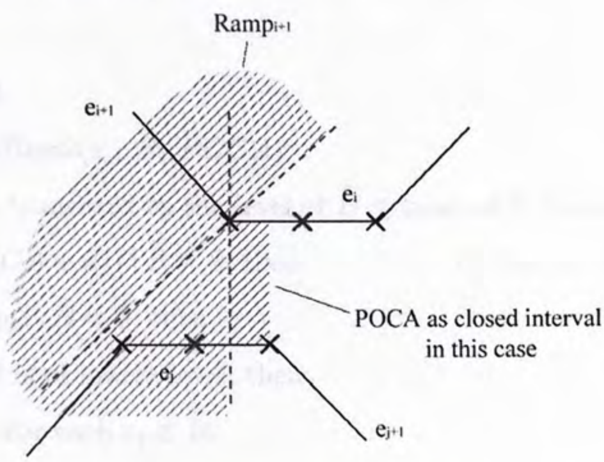


Figure 23b

With careful examination, this approach, hopefully, will solve the problem. But indeed, there is a much simpler method that can solve this problem immediately.

<u>Approach 3 : By enlarging $Ramp_i$</u>

The core of the problem is that even $y \in Ramp_i$, $\widetilde{e_j}$ may not be in $Ramp_i$. So, by enlarging $Ramp_i$ appropriately, $\widetilde{e_j}$ will be included.

Let $d \triangleq max\ \{||e_i||\}$

Let $\widetilde{Ramp_i} \triangleq \overline{N(Ramp_i; \frac{d}{2})}$

Moreover, in step 3 : $POCA$ computation, the statement "$N(\widetilde{e_i}; POCA) \cap \widetilde{B} \neq \emptyset$" has to be replaced by something similar to offset the effect of identifying edges with their mid-points.

Modified step 3 : $POCA$ computation

Procedure FindPOCA2()

$POCA := 2 * minRad(P_n)$

For i = 1 to n,

CheckBox2(e_i, B, $POCA$)

Next i

End FindPOCA2

Procedure CheckBox2(e_i, $\widetilde{B}$, $POCA$)

If octal tag of e_i truncated to the level of $\widetilde{B} \leq$ label of $\widetilde{B}$, then

If $N(\widetilde{e_i}; POCA + d) \cap \widetilde{B} \neq \emptyset$, then ($\frac{d}{2}$ from e_i and $\frac{d}{2}$ from e_j)

If $\widetilde{Ramp_i} \cap \widetilde{B} \neq \emptyset$, then

If $\widetilde{B}$ is of lowest level, then

For each $e_j \in \widetilde{B}$

Check e_i, e_j and update $POCA$ if necessary

Next e_j

Else

For each child box $\widetilde{\widetilde{B}}$ of $\widetilde{B}$

CheckBox2(e_i, $\widetilde{\widetilde{B}}$, $POCA$)

Next $\widetilde{\widetilde{B}}$

End if

End if

End if

End if

End CheckBox2

Remark :

In scenario 1 ($C^{1,1}$ knot), there is another way to deal with d. That is to fix a certain small d in advance, then divides the knot as far as $d \geq max\ \{||e_i||\}$ This preparation will greatly improve the performance in the condition that $||e_i||$ are uneven.

In the natural situation, $\exists M, \forall i,\quad$ (the number of $\widetilde{e_j} \in Ramp_i) < M$ where M is independent of n. As mentioned before, the order of **Octrope** will be $O(n \log n)$ in this case. In the modified version, the crucial part is to replace $Ramp_i$ by $\widetilde{Ramp_i}$. Although it is not guaranteed, the number of $\widetilde{e_j} \in \widetilde{Ramp_i}$ should be less than $2M$. As a result, the order will still be $O(n \log n)$. So, this modification actually solve the problem.

<u>**Generalization :**</u>

Although all computations and discussions in this chapter are concerning P_n as a knot, a slight modification on the definitions and algorithms will extend its scope to P_n as link.

Chapter 5

Arc presentation

There are three sections in this chapter : definitions, basic theorems, ropelength upper bound. The first section is just a detailed definition of an arc presentation and arc-index.

The existence of arc presentation for a given tame link type is proved in detail in section 2. As a result, the arc-index of a link type can be defined. In [19], there is an upper bound of arc-index by crossing number. Since, with arc presentation given, a link with ropelength bounded by arc-index can be constructed quite directly, an upper bound of ropelength of a given link type by crossing number can be made. The construction of such a link will be described in depth in the last section. The major conclusion of this chapter is that the ropelength of a link type is bounded above by $Cross([L])^2$.

5.1 Definitions

Notations :

Z	$\{(x,y,z) \in \mathbb{R}^3 : \quad x = y = 0\}$ (z-axis)
H_{θ_0}	$\{(r,\theta,z) \in \mathbb{R}^3 : \quad \theta = \theta_0\}$ (in polar coordinates)
$int(X)$	interior of set X
$Cross([L])$	crossing number of $[L]$

Definition 5.1.1:

A is called an arc presentation of a link L if

$A \in [L]$ *and*

$\exists n \in \mathbb{Z}^+, \quad A \subseteq \bigcup_{i=1}^{n} H_{\frac{2\pi i}{n}}$ *and* $A \cap Z = 0 \times 0 \times \{1, 2, \cdots, n\}$ *and*

$(\forall 1 \le i \le n, \quad \overline{A \cap int(H_{\frac{2\pi i}{n}})}$ *is a simple arc and*

$\overline{A \cap int(H_{\frac{2\pi i}{n}})} \cap Z$ *are two distinct points*),

$Arc(A) \triangleq n$ is called the arc-index of A

Example 5.1.1:

Trefoil has an arc presentation with arc-index 5. (See figure 24)

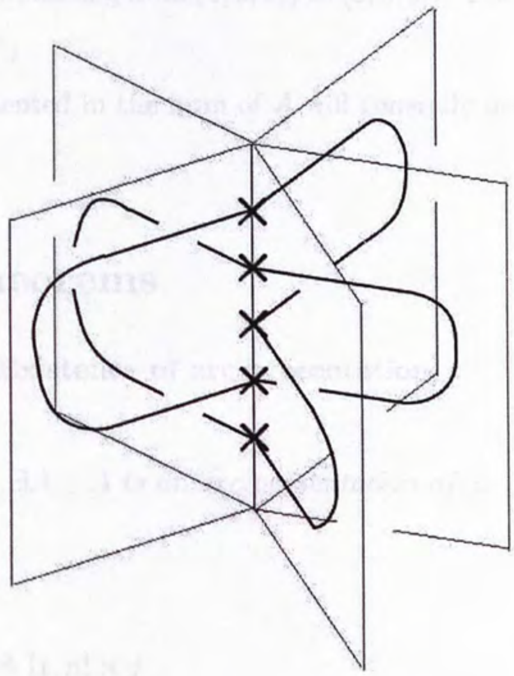


Figure 24

Remark :

In fact, an equivalent definition may be used instead.

A called an arc presentation of a link L if

$A \in [L]$ *and*

$\exists n \in \mathbb{Z}^+, \exists \theta_1, \theta_2, \cdots, \theta_n$ *distinct angles,*

$A \subseteq \bigcup_{i=1}^{n} H_{\theta_i}$ *and* $|A \cap Z| = n$ *and*

$(\forall 1 \leq i \leq n,\quad \overline{A \cap int(H_{\theta_i})}$ *is a simple arc and*

$\overline{A \cap int(H_{\theta_i})} \cap Z$ *are two distinct points)*

Certainly, by scaling the z-axis and rotating each H_{θ_i}, this arc-presentation can be transformed back to our definition.

For a given arc-presentation A, a set of 3-tuples can be obtained to uniquely describe A.

First of all, A is oriented. Then, starting from $i = 1$, $\exists x_i \neq y_i$, $\overline{A \cap int(H_{\frac{2\pi i}{n}})}$ is an oriented simple arc starting from $(0,0,x_i)$ to $(0,0,y_i)$. Then, the corresponding 3-tuple is $(x_i, y_i, \frac{2\pi i}{n})$.

Obviously, L represented in the form of A will generally not in C^1.

5.2 Basic Theorems

Theorem 5.2.1[18] (Existence of arc presentation) :

$$\forall L, \exists A, \quad A\ is\ an\ arc\ presentation\ of\ L$$

Proof[18]

$n \in \mathbb{Z}^+$

$C_i^n \triangleq i \times [1,n], R_j^n \triangleq [1,n] \times j$

$s \subseteq \mathbb{R}^2$ is called a stick if

s is a line segment st. $||s|| = 1$ and both end-points are in $\mathbb{Z} \times \mathbb{Z}$.

$S^n \triangleq \{G^n = \bigcup_{i=1}^m s_i\ st.\ s_i\ is\ a\ stick\ in\ (\bigcup_{i=1}^n C_i^n) \cup (\bigcup_{j=1}^n R_j^n):$

(Condition 1.) $\forall (i,j) \in G^n, \quad (i,j)\ is\ of\ type\ 2\ or\ 4,$

(Condition 2.) $\forall i, \quad G^n \cap C_i^n\ is\ the\ union\ of\ a\ line\ segment\ and\ isolated\ points$

$\forall j, \quad G^n \cap R_j^n\ is\ the\ union\ of\ a\ line\ segment\ and\ isolated\ points\ \}$

(See figure 25a)

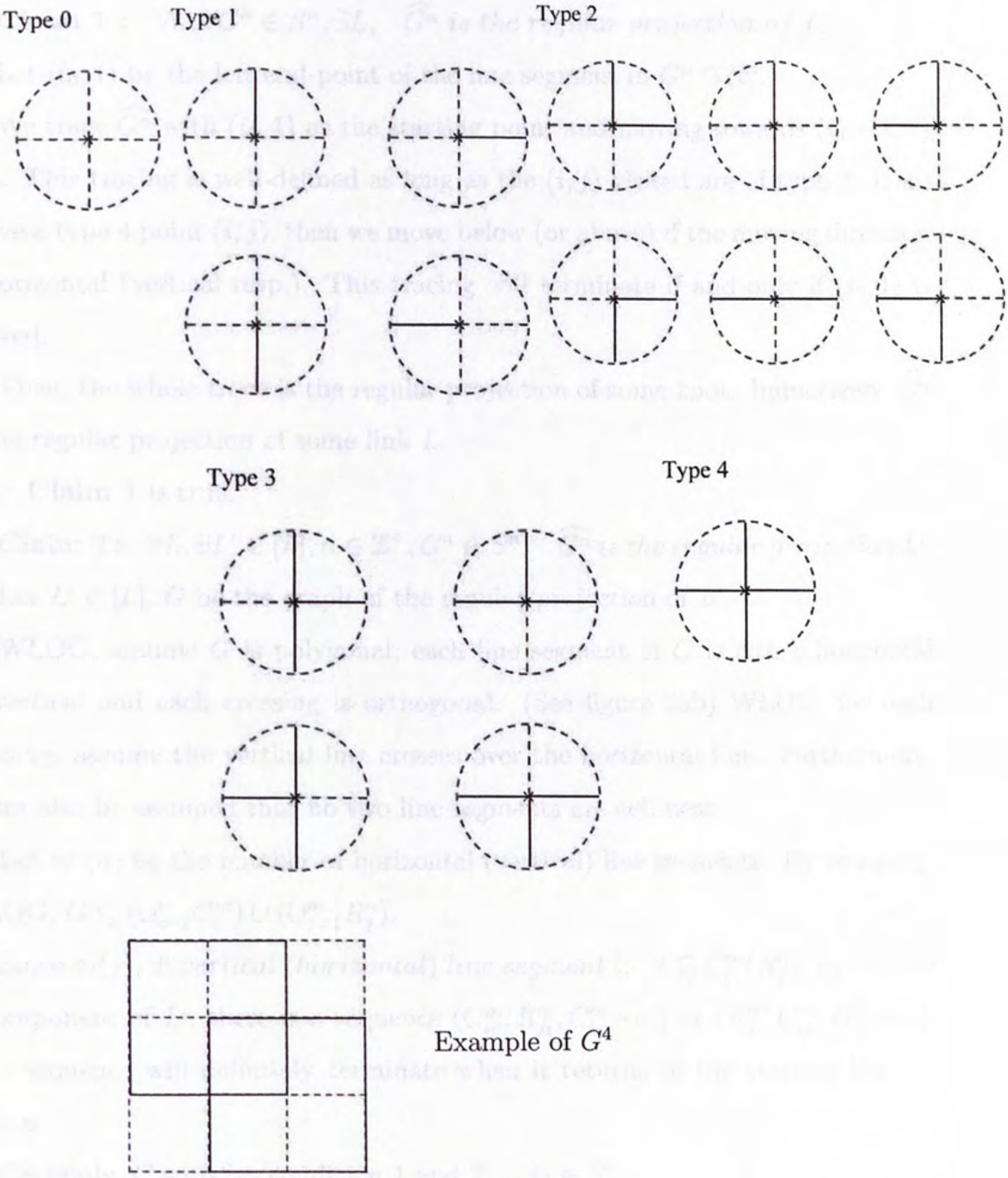


Figure 25a

$G^n \in S^n$,

Let $\widehat{G^n}$ be the diagram of G^n by transforming each $(i, j) \in G^n$ of type 4 st. the vertical part crosses over the horizontal part.

Claim 1 : $\forall n, \forall G^n \in S^n, \exists L,$ $\widehat{G^n}$ *is the regular projection of* L.

Let $(i_0, 1)$ be the left end-point of the line segment in $G^n \cap R^n_1$.

We trace $\widehat{G^n}$ with $(i_0, 1)$ as the starting point and moving towards $(i_0 + 1, 1)$ first. This tracing is well-defined as long as the (i, j) visited are of type 2. If we arrive a type 4 point (i, j), then we move below (or above) if the moving direction is horizontal (vertical resp.). This tracing will terminate if and only if $(i_0, 1)$ is arrived.

Then, the whole trace is the regular projection of some knot. Inductively, $\widehat{G^n}$ is the regular projection of some link L.

$\therefore$ **Claim 1** is true.

Claim 2 : $\forall L, \exists L' \in [L], n \in \mathbb{Z}^+, G^n \in S^n,$ $\widehat{G^n}$ *is the regular projection* L'.
Let $L' \in [L]$, G be the graph of the regular projection of L'.

WLOG, assume G is polygonal, each line segment of G is either horizontal or vertical and each crossing is orthogonal. (See figure 25b) WLOG, for each crossing, assume the vertical line crosses over the horizontal line. Furthermore, it can also be assumed that no two line segments are collinear.

Let m (n) be the number of horizontal (vertical) line segments. By rescaling, WLOG, $G \subseteq (\cup_{i=1}^n C_i^m) \cup (\cup_{j=1}^m R_j^n)$.

Since $\forall i(j), \exists!$ *vertical (horizontal) line segment* l, $l \subseteq C_i^m(R_j^n)$, by tracing a component of L', there is a sequence $(C^m_{i_1}, R^n_{j_1}, C^m_{i_2}, \cdots)$ or $(R^n_{j_1}, C^m_{i_1}, R^n_{j_2}, \cdots)$. This sequence will definitely terminate when it returns to the starting line. $\therefore$ $m = n$.

Certainly, G satisfies condition 1 and 2. $\therefore G \in S^n$.

$\therefore$ **Claim 2** is true.

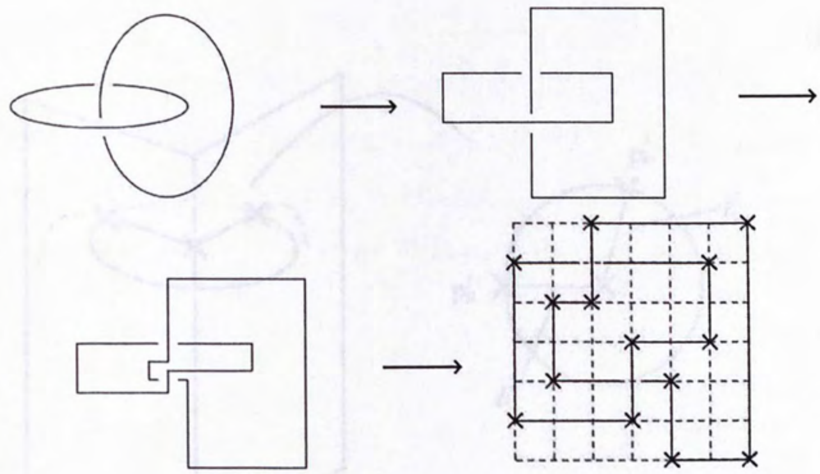


Figure 25b

Claim 3 : $\forall A$ *arc presentation with arc* $-$ *index* $n, \exists G^n \in S^n, \quad \widehat{G^n} \in [A]$.

Let $\epsilon \in \mathbb{R}^+$

WLOG, assume $A \cap \overline{N(Z;\epsilon)}$ is consists of n components where each component is the union of 2 connecting line segments $l^j_{j_1}, l^j_{j_2}$ st. $||l^j_{j_1}|| = ||l^j_{j_2}|| = \epsilon,\ l^j_{j_1}, l^j_{j_2} \subseteq \mathbb{R}^2 \times j,\ l^j_{j_1} \subseteq H_{\frac{2\pi j_1}{n}}$ *and* $l^j_{j_2} \subseteq H_{\frac{2\pi j_2}{n}}$. (See figure 25c)

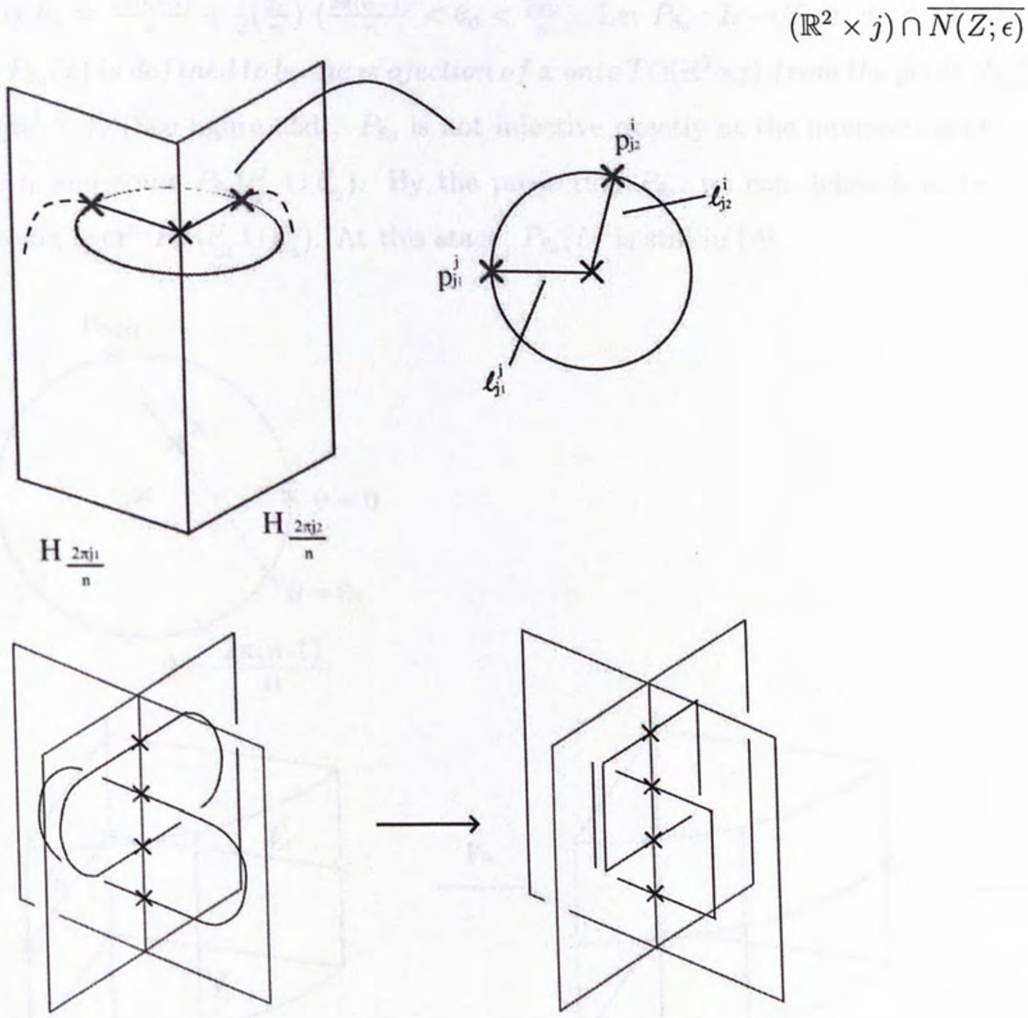


Figure 25c

$T \triangleq \partial\overline{N(Z;\epsilon)}$, $p^j_{j_1} \triangleq T \cap l^j_{j_1}$, $p^j_{j_2} \triangleq T \cap l^j_{j_2}$. Now, $\forall i, \exists j', j''$, $p^{j'}_i, p^{j''}_i \in H_{\frac{2\pi i}{n}}$ (and indeed in $H_{\frac{2\pi i}{n}} \cap T$). Link $p^{j'}_i$ and $p^{j''}_i$ with a line segment $\widetilde{l}_i$ to form a link L. Then, evidently, $L \in [A]$.

Fix $\theta_0 \triangleq \frac{2\pi(n-1)}{n} + \frac{1}{2}(\frac{2\pi}{n})$ $(\frac{2\pi(n-1)}{n} < \theta_0 < \frac{2\pi n}{n})$. Let $P_{\theta_0} : L \to T$ *st.* $\forall x \in l^j_{j_1} \cup l^j_{j_2}$, $P_{\theta_0}(x)$ *is defined to be the projection of* x *onto* $T \cap (\mathbb{R}^2 \times j)$ *from the point* $H_{\theta_0} \cap T \cap (\mathbb{R}^2 \times j)$ (See figure 25d). P_{θ_0} is not injective exactly at the intersection of some $\widetilde{l_i}$ and some $P_{\theta_0}(l^j_{j_1} \cup l^j_{j_2})$. By the projection P_{θ_0}, we can define $\widetilde{l_i}$ to be "crossing over" $P_{\theta_0}(l^j_{j_1} \cup l^j_{j_2})$. At this stage, $P_{\theta_0}(L)$ is still in $[A]$.

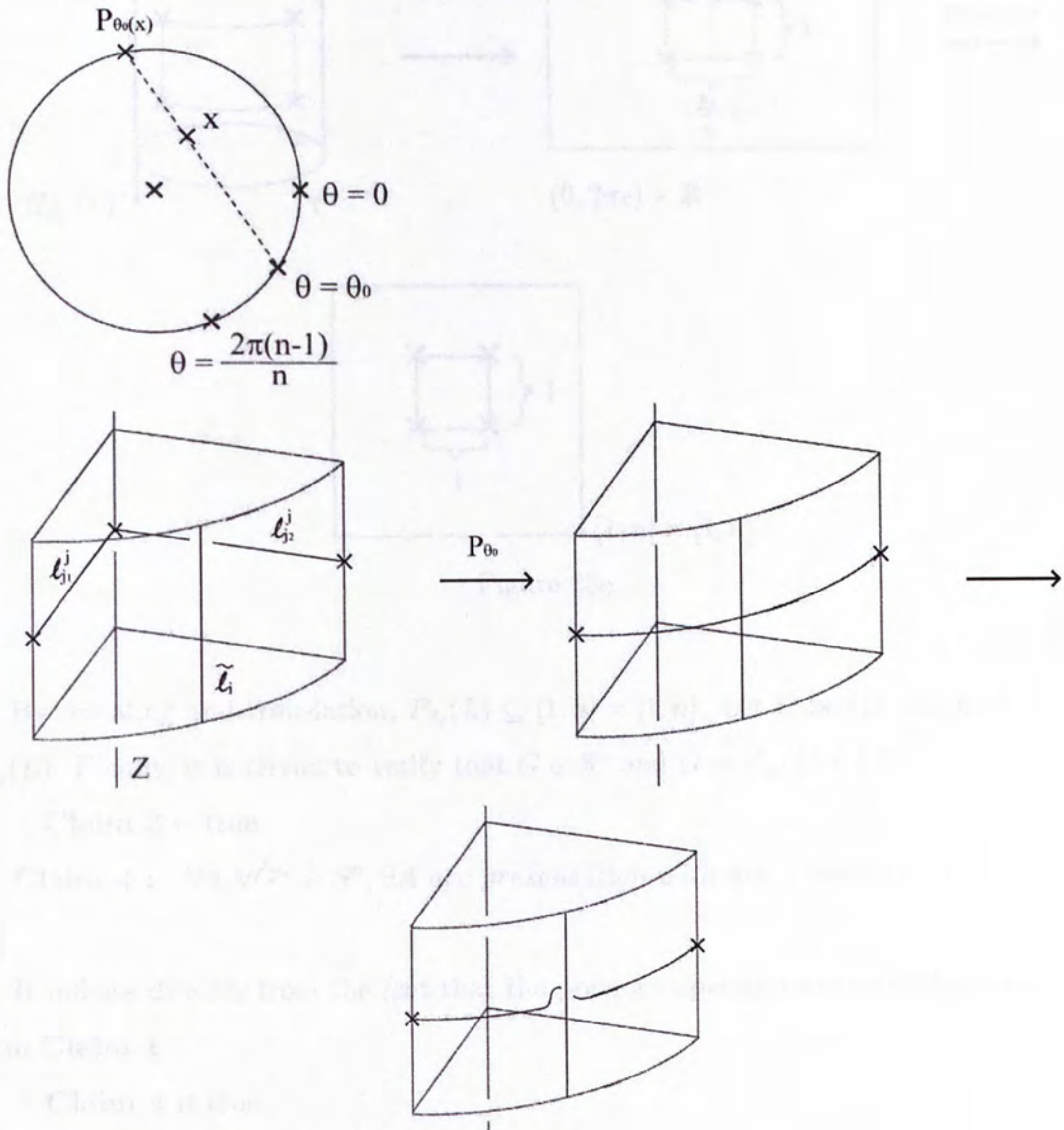


Figure 25d

Considering T as a tube and $P_{\theta_0}(L)$ as a diagram on T, by "cutting" T along the line $H_{\theta_0} \cap T$, T can be regarded as $(0, 2\pi\epsilon) \times \mathbb{R}$ and $P_{\theta_0}(L)$ will then be a diagram on it. (See figure 25e)

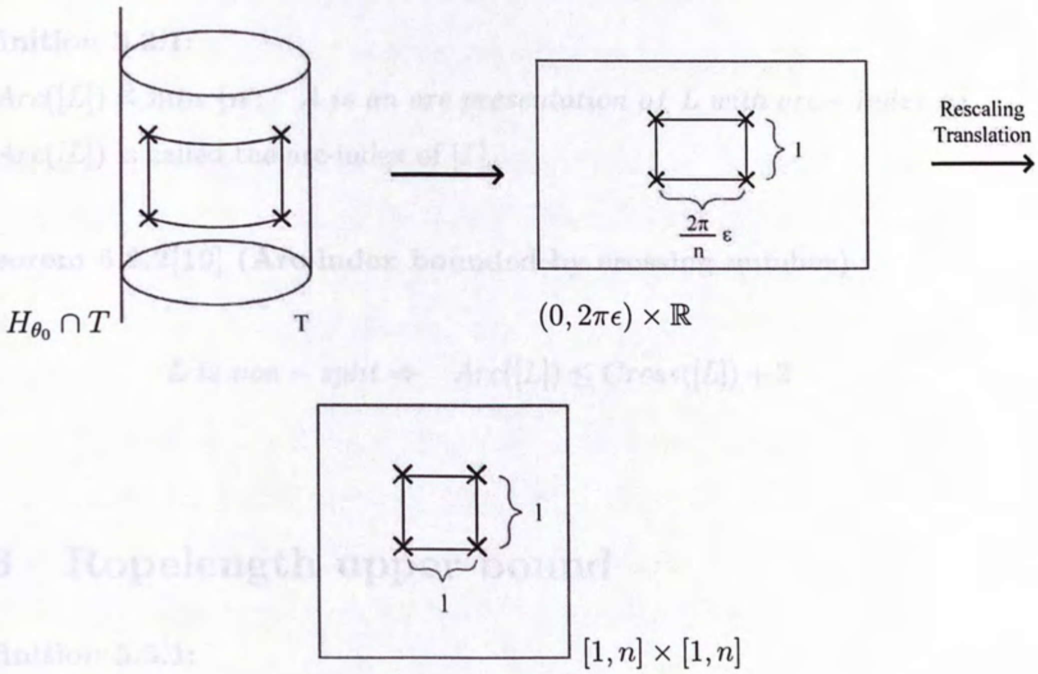


Figure 25e

By rescaling and translation, $P_{\theta_0}(L) \subseteq [1, n] \times [1, n]$. Let G be the graph of $P_{\theta_0}(L)$. Finally, it is trivial to verify that $G \in S^n$ and $\widehat{G} = P_{\theta_0}(L) \in [A]$.

$\therefore$ **Claim 3** is true.

Claim 4 : $\forall n, \forall G^n \in S^n, \exists A$ *arc presentation with arc* $-$ *index* n, $\widehat{G^n} \in [A]$.

It follows directly from the fact that the previous operation is reversible and from **Claim 1**.

$\therefore$ **Claim 4** is true.

By **Claim 2 and 4**, the proof is complete.

Remark :

According to **Claim 3 and 4** in the proof, $\widehat{G^n}$ is just a representation of A. It is called the loop and lines diagram of A.

Definition 5.2.1:

$Arc([L]) \triangleq min\ \{n: \quad A\ is\ an\ arc\ presentation\ of\ L\ with\ arc-index\ n\}$,

$Arc([L])$ is called the arc-index of $[L]$

Theorem 5.2.2[19] (Arc-index bounded by crossing number) :

$$L\ is\ non-split \Rightarrow \quad Arc([L]) \leq Cross([L])+2$$

5.3 Ropelength upper bound

Definition 5.3.1:

$A\ is\ an\ arc\ presentation\ with\ Arc(A) = n\ and\ corresponding\ 3\text{-}tuples\ (x_i, y_i, \frac{2\pi i}{n})$

$Skip(A) \triangleq \sum_{i=1}^{n} |x_i - y_i|$

Theorem 5.3.1[8] (Bound by arc-index) :

$$A\ is\ an\ arc\ presentation\ with\ Arc(A) = n$$

$$\Rightarrow \quad \exists L \in [A], \quad RL(L) \leq \frac{2n}{\tan\frac{\pi}{n}} + (\pi-2)n + 2Skip(A)$$

Proof[8]

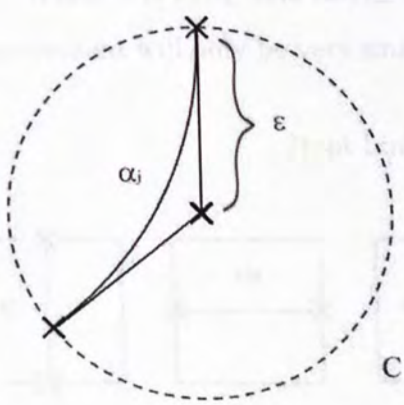
The proof is just a construction of L st. $R(L) = \frac{1}{2}$, $L(L) \leq \frac{1}{2}[\frac{2n}{\tan\frac{\pi}{n}} + (\pi - 2)n + 2Skip(A)]$.

First of all, $\epsilon \triangleq \frac{1}{2\tan\frac{\pi}{n}}$. WLOG, let L to be the link in **Claim 3** of the proof of **Theorem 5.2.1**. Using the same notations in **Claim 3**, $\forall j$, $l^j_{j_1}, l^j_{j_2}$ are 2 connecting line segments in $\mathbb{R}^2 \times j$ *st.* $||l^j_{j_1}|| = ||l^j_{j_2}|| = \epsilon$. We transform these 2 line segments into a single arc α_j. (See figure 26a) $\forall i$, $\widetilde{l_i}$ is a line segment with end-points $p_i^{j'}$ in level j' and $p_i^{j''}$ in level j''. We transform $\widetilde{l_i}$ into the union of 2 arcs and 1 line segment β_i. (See figure 26b) This finishes the construction.

Apparently, $L \in [A]$. In addition, $\forall j$, $||\alpha_j|| \leq 2\epsilon$. It is also obvious that $\sum_{i=1}^n ||\beta_i|| = \sum_{i=1}^n(\frac{\pi}{2} + |x_i - y_i| - 1) = (\frac{\pi}{2} - 1)n + Skip(A)$. Consequently, $L(L) \leq \frac{n}{\tan\frac{\pi}{n}} + (\frac{\pi}{2} - 1)n + Skip(A)$. From the construction, it can also be deduced that the injectivity radius $R(L) = \frac{1}{2}$ $(L \in C^{1,1})$.

$\because R(L) = \frac{1}{2}$, $L(L) \leq \frac{1}{2}[\frac{2n}{\tan\frac{\pi}{n}} + (\pi - 2)n + 2Skip(A)]$,

$\therefore RL(L) \leq \frac{2n}{\tan\frac{\pi}{n}} + (\pi - 2)n + 2Skip(A)$.



α_j is an arc of a circle

α_j is orthogonal to the circle C at both end-points

Figure 26a

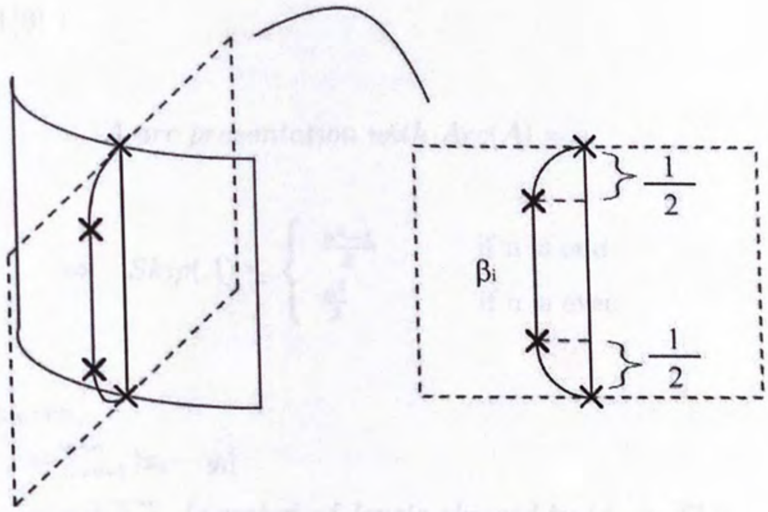
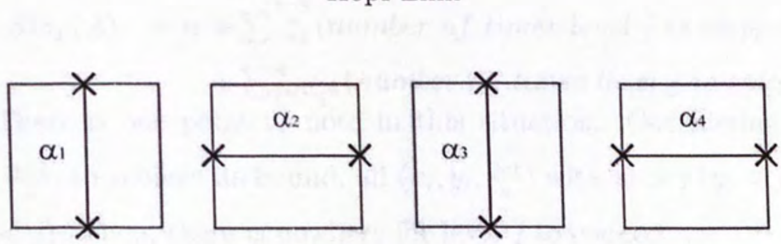


Figure 26b

Remark :

It can be seen that the bound given by $||\alpha_j||$ can not be sharp when n is odd because α_j can never be the diameter and the length of the arc will strictly less than 2ϵ. When n is even, this bound is achievable. (See figure 27) In any cases, the improvement will only be very small and is meaningless concerning the order.

Hopf Link



All are diameters

Figure 27

Lemma 5.3.1[8] :

$$A \textit{ arc presentation with } Arc(A) = n$$

$$\Rightarrow \quad Skip(A) \leq \begin{cases} \frac{n^2-1}{2} & \text{if n is odd} \\ \frac{n^2}{2} & \text{if n is even} \end{cases}$$

Proof[8]

When n is even,

$$\begin{aligned} Skip(A) &= \sum_{i=1}^{n} |x_i - y_i| \\ &= n + \sum_{i=1}^{n}(number\ of\ levels\ skipped\ by\ (x_i, y_i, \tfrac{2\pi i}{n})) \\ &= n + \sum_{j=1}^{n}(number\ of\ times\ level\ j\ is\ skipped) \\ &= n + \sum_{j=1}^{\frac{n}{2}}(number\ of\ times\ level\ j\ is\ skipped) \\ &\quad + \sum_{j=\frac{n}{2}+1}^{n}(number\ of\ times\ level\ j\ is\ skipped) \end{aligned}$$

When level j is skipped by $(x_i, y_i, \frac{2\pi i}{n})$, then ($x_i > j$ and $y_i < j$) or ($x_i < j$ and $y_i > j$). In addition, each level $j, \exists! i_1 \neq i_2, \quad j = x_{i_1} = y_{i_2}$. (one goes in and one goes out)

$\therefore Skip(A) \leq n + \sum_{j=1}^{\frac{n}{2}} 2(j-1) + \sum_{j=\frac{n}{2}+1}^{n} 2(n-j) = \frac{n^2}{2}$.

When n is odd,

Similarly,

$$\begin{aligned} Skip(A) &= n + \sum_{j=1}^{\frac{n-1}{2}}(number\ of\ times\ level\ j\ is\ skipped) \\ &\quad + \sum_{j=\frac{n+1}{2}}^{n}(number\ of\ times\ level\ j\ is\ skipped) \end{aligned}$$

There is one point to note in this situation. Considering the middle level $j = \frac{n+1}{2}$, to achieve its bound, all $(x_i, y_i, \frac{2\pi i}{n})$ with $x_i > j$ ($y_i > j$) will have $y_i < j$ ($x_i < j$). Then, there is nowhere for level j to connect to.

$\therefore$ number of times level $\frac{n+1}{2}$ is skipped $\leq 2(\frac{n-1}{2}) - 1 = n - 2$.

$\therefore Skip(A) \leq n + \sum_{j=1}^{\frac{n-1}{2}} 2(j-1) + \sum_{j=\frac{n+3}{2}}^{n} 2(n-j) + n - 2 = \frac{n^2-1}{2}$.

Remark :

This bound is sharp in the following sense.

$\forall n, \exists A$ $st.$ $Arc(A) = n,$

$$Skip(A) = \begin{cases} \frac{n^2-1}{2} & \text{if n is odd} \\ \frac{n^2}{2} & \text{if n is even} \end{cases}$$

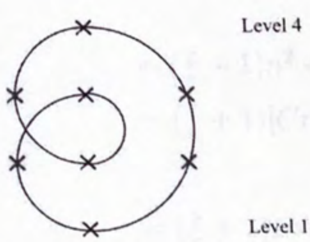
For n is even, this A is defined by

$(n, \frac{n}{2}, \frac{2\pi}{n}), (\frac{n}{2}, n-1, \frac{2\pi(2)}{n}), (n-1, \frac{n}{2}-1, \frac{2\pi(3)}{n}), (\frac{n}{2}-1, n-2, \frac{2\pi(4)}{n}), \cdots,$
$(\frac{n}{2}+1, 1, \frac{2\pi(n-1)}{n}), (1, n, \frac{2\pi(n)}{n})$. (See figure 28)

For n is odd, this A is defined by

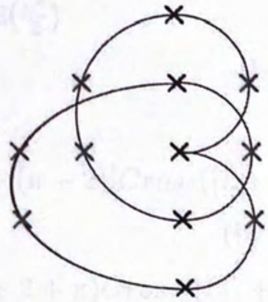
$(n, \frac{n-1}{2}, \frac{2\pi}{n}), (\frac{n-1}{2}, n-1, \frac{2\pi(2)}{n}), (n-1, \frac{n-1}{2}-1, \frac{2\pi(3)}{n}), (\frac{n-1}{2}-1, n-2, \frac{2\pi(4)}{n}), \cdots,$
$(1, \frac{n+1}{2}, \frac{2\pi(n-1)}{n}), (\frac{n+1}{2}, n, \frac{2\pi(n)}{n})$.

$n = 4$



$Skip(A) = 4 + 4 = \frac{4^2}{2}$

$n = 5$



$Skip(A) = 5 + 7 = \frac{5^2-1}{2}$

Figure 28

Theorem 5.3.2[8] (Bound by crossing number) :

$$L \, is \, non-split$$

$$\Rightarrow \quad RL([L]) \leq (\frac{2}{\pi}+1)Cross([L])^2+(\frac{8}{\pi}+2+\pi)Cross([L])+(\frac{8}{\pi}+\frac{4\pi}{3})$$

Proof[8]

By Laurent expansion, $\forall x \in (0,\frac{\pi}{2}], \quad \cot x \leq \frac{1}{x}-\frac{x}{3}$

Let A be an arc presentation of L st. $Arc(A)=Arc([L])$,

Let L' constructed from A by the construction described in the proof of **Theorem 5.3.1**.

By **Theorem 5.3.1**,

$$\begin{aligned} RL(L') &\leq \tfrac{2n}{\tan\frac{\pi}{n}}+(\pi-2)n+2Skip(A) \\ &\leq 2n(\tfrac{n}{\pi}-\tfrac{\pi}{3}\tfrac{1}{n})+(\pi-2)n+2Skip(A) && (n\geq 2) \\ &\leq 2n(\tfrac{n}{\pi}-\tfrac{\pi}{3}\tfrac{1}{n})+(\pi-2)n+2(\tfrac{n^2}{2}) \end{aligned}$$

(by **Lemma 5.3.1**)

$$\begin{aligned} &=(\tfrac{2}{\pi}+1)n^2+(\pi-2)n-\tfrac{2\pi}{3} \\ &=(\tfrac{2}{\pi}+1)[Cross([L])+2]^2+(\pi-2)[Cross([L])+2]-\tfrac{2\pi}{3} \end{aligned}$$

(by **Theorem 5.2.2**)

$$=(\tfrac{2}{\pi}+1)Cross([L])^2+(\tfrac{8}{\pi}+2+\pi)Cross([L])+(\tfrac{8}{\pi}+\tfrac{4\pi}{3})$$

Lemma 5.3.2[8] :

$$L \, is \, prime \Rightarrow \quad L \, is \, non-split$$

Proof[8]

Assume L is split,

$L \triangleq L_1 \cup L_2$ where L_1, L_2 are non-trivial, separated components

Let K be an unknot.

$L = L_1 \cup L_2 = L_1 \# (K \cup L_2)$ where the connected sum is done on L_1 and K.

$\because L_1,\ K \cup L_2$ are non-trivial,

$\therefore L$ is not prime.

Remark :

So, in particular, **Theorem 5.3.2** implies that

$$L \textit{ is prime}$$

$$\Rightarrow \quad RL([L]) \leq (\frac{2}{\pi}+1)Cross([L])^2 + (\frac{8}{\pi}+2+\pi)Cross([L]) + (\frac{8}{\pi}+\frac{4\pi}{3})$$

Theorem 5.3.3[8] (Bound by crossing number) :

$$L \textit{ is a non}-\textit{split link with prime components } L_1, L_2, \cdots, L_n$$

$$\Rightarrow \quad RL([L]) \leq (\frac{2}{\pi}+1)\sum_{i=1}^{n} Cross([L_i])^2 + (\frac{8}{\pi}+2+\pi)\sum_{i=1}^{n} Cross([L_i]) + (\frac{8}{\pi}+2+\frac{\pi}{3})n + (\pi-2)$$

Proof[8]

For $n = 2$:

Let A_1 (A_2) be an arc presentation of L_1 (L_2) st. $Arc(A_1) = Arc([L_1])$ $(Arc(A_2) = Arc([L_2]))$.

Let L_1' (L_2') constructed from A_1 (A_2) by the construction described in the proof of **Theorem 5.3.1**.

Claim 1 : $\exists L' \in [L'_1 \# L'_2], \quad RL(L') \leq RL(L'_1) + RL(L'_2) - (\pi - 2)$.

Considering the "bottom part" of L'_1 ($L'_1 \cap (\mathbb{R}^2 \times [1, \frac{3}{2}])$), it is the union of a quarter circular arc γ_1^1, a circular arc γ_2^1 and another quarter circular arc γ_3^1. (See figure 29a) Let T_1^1 (T_3^1) be the line st. $T_1^1 // Z$ ($T_3^1 // Z$) and T_1^1 (T_3^1) is a tangent of γ_1^1 (γ_3^1).

The first step is to transform L'_1 by rotating γ_1^1 about T_1^1, γ_3^1 about T_3^1 and straightening γ_2^1. Embed L'_2 in $\mathbb{R}^2 \times [-n, -1]$ so that the top level of L'_2 corresponds to -1. Do the same operation on the top level of L'_2 to obtain $\gamma_1^2, \gamma_2^2, \gamma_3^2, T_1^2, T_3^2$.

The second step is to move L'_2 so that T_1^1, T_1^2 collinear, γ_2^2 collapsed in γ_2^1. $l_1 \triangleq ||\gamma_2^1||$ $l_2 \triangleq ||\gamma_2^2||$. (WLOG, assume $l_1 \geq l_2$)

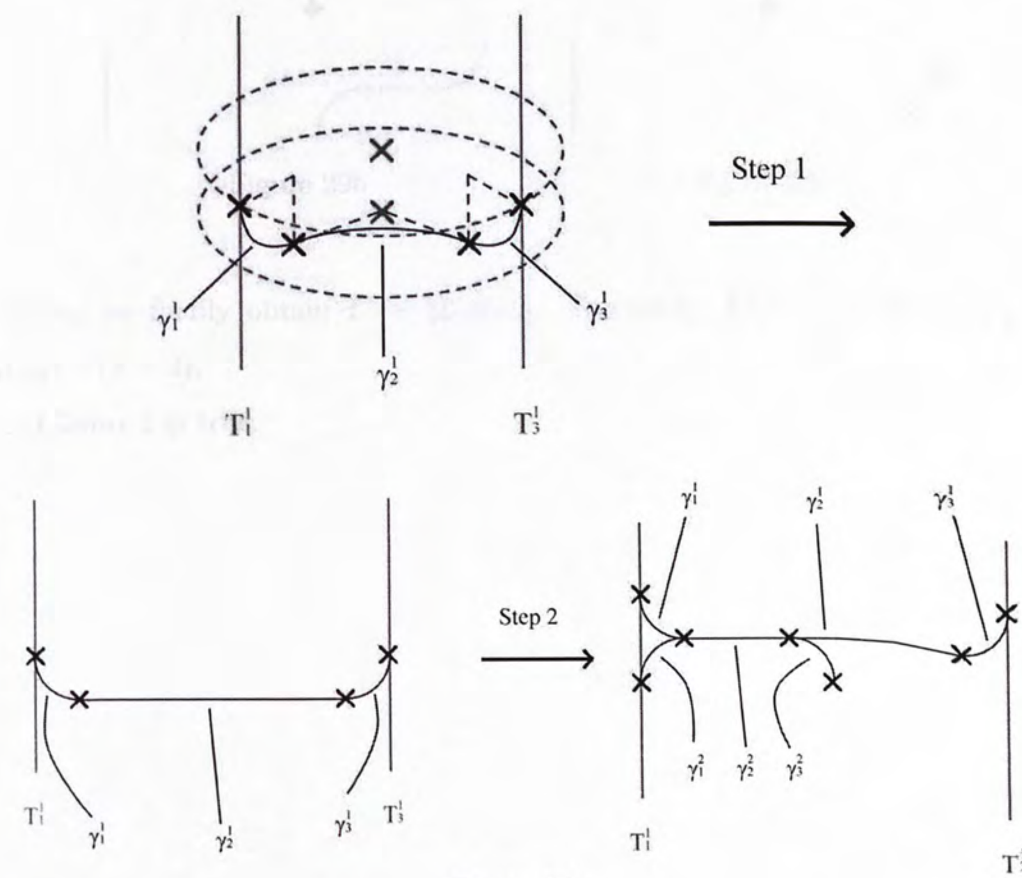


Figure 29a

The final step depends on l_1, l_2.

If $l_1 - l_2 \geq 1$, then

Connect L'_1, L'_2 as in figure 29b.

If $l_1 - l_2 < 1$, then

Connect L'_1, L'_2 as in figure 29c.

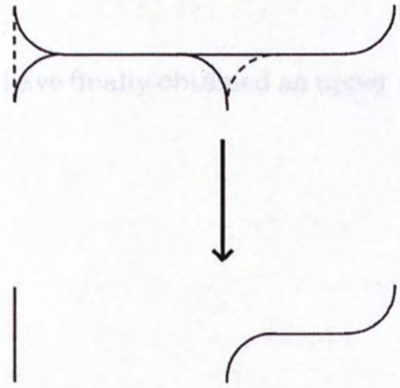


Figure 29b

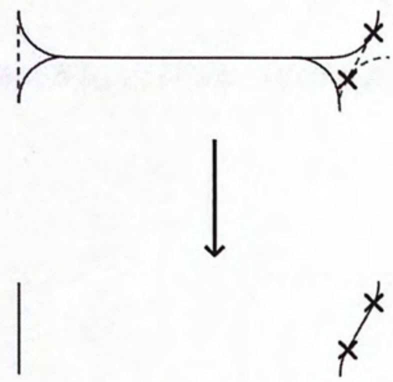


Figure 29c

Then, we finally obtain $L' \in [L'_1 \# L'_2]$. Obviously, $RL(L') \leq RL(L'_1) + RL(L'_2) - (\pi - 2)$.

$\therefore$ **Claim 1** is true.

Inductively, the same operation can be done for any n.

$\therefore \exists L' \in [L'_1 \# L'_2 \# \cdots L'_n]$,

$$
\begin{aligned}
RL([L]) \leq RL(L') &\leq \textstyle\sum_{i=1}^{n} RL(L'_i) - (n-1)(\pi - 2) \\
&\leq \textstyle\sum_{i=1}^{n} [(\frac{2}{\pi}+1) Cross([L_i])^2 + (\frac{8}{\pi}+2+\pi) Cross([L_i]) + \\
&\qquad\qquad (\tfrac{8}{\pi}+\tfrac{4\pi}{3})] - (n-1)(\pi-2) \\
&\qquad\qquad \text{(by } \textbf{Lemma 5.3.2} \text{ and the proof of } \textbf{Theorem 5.3.2}\text{)} \\
&= (\tfrac{2}{\pi}+1) \textstyle\sum_{i=1}^{n} Cross([L_i])^2 + (\frac{8}{\pi}+2+\pi) \sum_{i=1}^{n} Cross([L_i]) + \\
&\qquad\qquad (\tfrac{8}{\pi}+2+\tfrac{\pi}{3})n + (\pi-2)
\end{aligned}
$$

Remark :

So, we have finally obtained an upper bound of $RL([L])$ of order $O(Cross([L])^2)$.

Chapter 6

Hamiltonian knot projection

There are three sections in this chapter : Hamiltonian RPG, embedding of RPG, Ropelength upper bound. In the first section, the definitions of $k-regular$, *RPG* (regular projection graph), *connectivity* and *edge − connectivity* are given. A link is called *Hamiltonian* if there is a representative such that it has a *RPG* which is a *Hamiltonian* graph. A link is called *minimally Hamiltonian* if in addition, the number of vertices of this graph is just the crossing number of the link type. The major results of this section is that all link types are *Hamiltonian*, but not all is *minimally Hamiltonian*. In addition, the number of vertices of the *Hamiltonian* graph found can be assumed to be bounded above by 4 $Cross([L])$.

The whole section 2 is devoted to describe the procedure of embedding a *Hamiltonian RPG* into a rectangular box with unit-length-sticks only. This procedure will be explained step by step. The final product can be proved to be ambient isotopic to the original graph. Instead of giving a precise proof of this fact, a rough recovering process is given.

By considering all possible lengths of the embedded edges, an upper bound of the length of the embedded lattice graph by the number of vertices of the original graph can be induced. Furthermore, by smoothing each corner by a quarter circular arc, ropelength of the embedded lattice graph is well-defined. As a result, an upper bound of the ropelength of a link type by crossing number can be constructed. The major conclusion of this chapter is that the ropelength of a link type is bounded above by $Cross([L])^{\frac{3}{2}}$.

6.1 Hamiltonian RPG

Definition 6.1.1:

G is a planar multigraph,

$V(G) \triangleq \{v\ vertex\ of\ G\}$,

$E(G) \triangleq \{e\ edge\ of\ G\}$,

G is called $k-regular$

if $\forall v \in V(G), \quad deg(v) = k$,

G is called a $RPG(L)$ (regular projection graph of L)

if G is the graph of a regular projection of L where $V(G) = \{Crossing\ point\}$,

G is called a $minRPG(L)$

if G is a $RPG(L)$ and $|V(G)| = Cross([L])$

Remark :

Obviously, $\forall L, \forall G\ is\ a\ RPG(L), \quad G\ is\ 4-regular$. In addition, $\forall G\ is\ 4-regular, \exists L, \quad G\ is\ a\ RPG(L)$.

If G is a $RPG(L)$ and $\exists (v,v) \in E(G)$ (*loop*), then by a $R-move$, this v (and hence (v,v)) can be eliminated. (See figure 30)

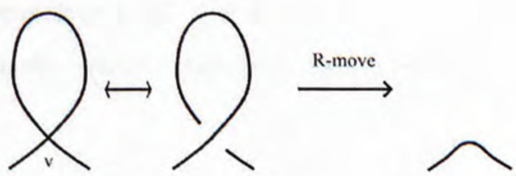


Figure 30

Hence, we have the following assumption in this chapter.

Assumption :

$$\forall L, \forall G \; is \; a \; RPG(L), \quad G \; has \; no \; loop$$

Definition 6.1.2:

G is a planar multigraph,

G is called $k-connected$ $(k \in \mathbb{Z}^+)$

if $|V(G)| \geq k+1$ *and* $(\forall X \subseteq V(G) \; st. \; |X| < k, \quad G \setminus X \; is \; connected)$,

G is called $k - edge - connected$ $(k \in \mathbb{Z}^+)$

if $\forall Y \subseteq E(G) \; st. \; |Y| < k, \quad G \setminus Y \; is \; connected$,

$Connectivity \; of \; G \triangleq max \; \{k : \quad G \; is \; k - connected\}$,

$Edge - connectivity \; of \; G \triangleq max \; \{k : \quad G \; is \; k - edge - connected\}$

Remark :

By deleting the 4 adjacent vertices of a fixed vertex of a $RPG(L)$ G, G becomes disconnected. So, $connectivity \; of \; G \leq 4$. Similarly, by deleting the 4 edges incident with a fixed vertex, G becomes disconnected. So, $edge-connectivity \; of \; G \leq 4$ as well.

Lemma 6.1.1[9] :

$$L \; is \; non-trivial \; and \; non-split, \; G \; is \; a \; minRPG(L)$$

$\Rightarrow \quad (a) \quad G \; is \; 2-connected$

$(b) \quad edge - connectivity \; of \; G = 2 \; or \; 4$

$(c) \quad L \; is \; prime \Rightarrow \quad edge - connectivity \; of \; G = 4$

Definition 6.1.3:

L is called *Hamiltonian*

if $\exists L' \in [L], \exists G\ a\ RPG(L'),\quad G\ is\ Hamiltonian,$

L is called *minimally Hamiltonian*

if $\exists L' \in [L], \exists G\ a\ Hamiltonian\ RPG(L'),\quad |V(G)| = Cross([L])$

Theorem 6.1.1[9] (Counterexample of minimally Hamiltonian) :

$$9_{35}\ is\ not\ minimally\ Hamiltonian$$

Remark :

In particular, 9_{35} is a prime knot. As a result of this theorem, we cannot assume $\forall L, \exists L' \in [L], \exists G\ minRPG(L'),\quad G\ is\ Hamiltonian$. Since we are focussing only on order, the goal of obtaining a *Hamiltonian* G can still be achieved by making a *Hamiltonian* graph G st. $|V(G)| \leq k\, Cross([L])$ for some constant k.

Theorem 6.1.2[9] (Connected sum of Hamiltonian) :

$L_1, L_2, \cdots, L_n$ *are Hamiltonian with* $L_i' \in L_i$ *and* G_i *is a Hamiltonian* $RPG(L_i')$

$$\Rightarrow \quad L_1 \# L_2 \# \cdots L_n\ is\ Hamiltonian\ with\ L' \in [L_1 \# L_2 \# \cdots L_n]\ and$$

$$G\ is\ a\ Hamiltonian\ RPG(L')\ st.\ |V(G)| = \sum_{i=1}^{n} |V(G_i)|$$

Theorem 6.1.3[20] :

$$G \ is \ 4-connected \Rightarrow \quad G \ is \ Hamiltonian$$

Theorem 6.1.4[9] :

$$L \ is \ prime$$

$$\Rightarrow \quad \exists L' \in [L], \exists G \ a \ Hamiltonian \ RPG(L'), \quad |V(G)| \leq 4 \ Cross([L])$$

Theorem 6.1.5[9] :

$$\forall L, \exists L' \in [L], \exists G \ a \ Hamiltonian \ RPG(L'), \quad |V(G)| \leq 4 \ Cross([L])$$

6.2 Embedding of RPG

Let G be a *Hamiltonian* $RPG(L)$ with *Hamiltonian cycle* C

Let $v_1, v_2, \cdots, v_n$ be the vertices ordered by C

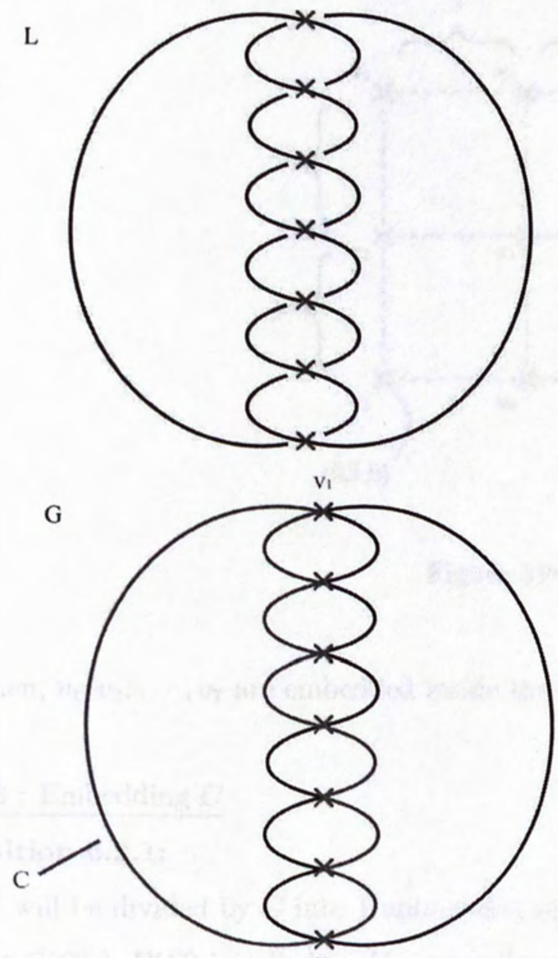
Let $k \triangleq \lceil \sqrt{n} \rceil$

A precise systematic algorithm for embedding G into a lattice graph F is described in the paper "*Hamiltonian knot projections and lengths of thick knots*" [9].

Instead of giving the precise algorithm, an example illustrating the steps will be given.

Example 6.2.1:

Figure 31a



7 crossing points

$k = \lceil \sqrt{7}\, \rceil = 3$

Step 1 : Embedding the vertices

$v_1, v_2, \cdots, v_7$ are embedded in the $z = 0$ plane as in figure 31b.

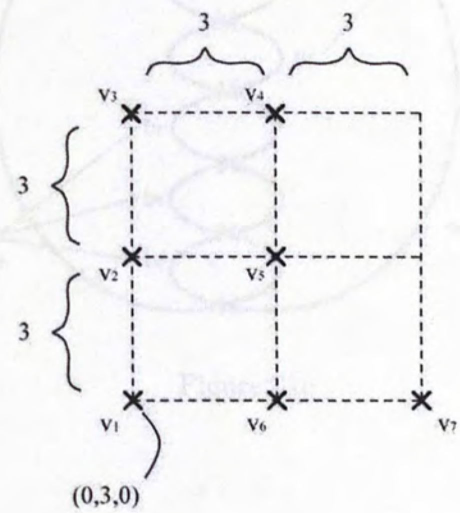


Figure 31b

Then, $v_1, v_2, \cdots, v_7$ are embedded inside the square $[0, 6] \times [3, 9] \times 0$.

Step 2 : Embedding C

Definition 6.2.1:

$\mathbb{R}^2$ will be divided by C into 1 unbounded region and 1 bounded region,

$e \in E(G) \setminus E(C)$ is called an $U - edge$ if e is inside the unbounded region

$e \in E(G) \setminus E(C)$ is called a $B - edge$ if e is inside the bounded region (See figure 31c)

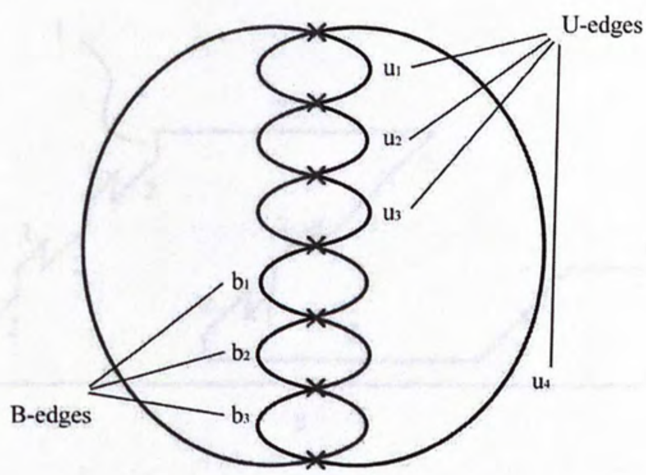


Figure 31c

Definition 6.2.2:

$e \neq e'$ incident with v_i st. $e, e' \notin C$,

v_i is called *type*(a) if

$(e\ is\ an\ U - edge\ and\ e'\ is\ a\ B - edge)\ or$

$(e\ is\ a\ B - edge\ and\ e'\ is\ an\ U - edge)$,

v_i is called *type*(b) if

$e, e'\ are\ both\ U - edges$,

v_i is called *type*(c) if

$e, e'\ are\ both\ B - edges$

$(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ are of *type* ($(b), (b), (b), (a), (c), (c), (a)$)

C will then be embedded as in figure 31d.

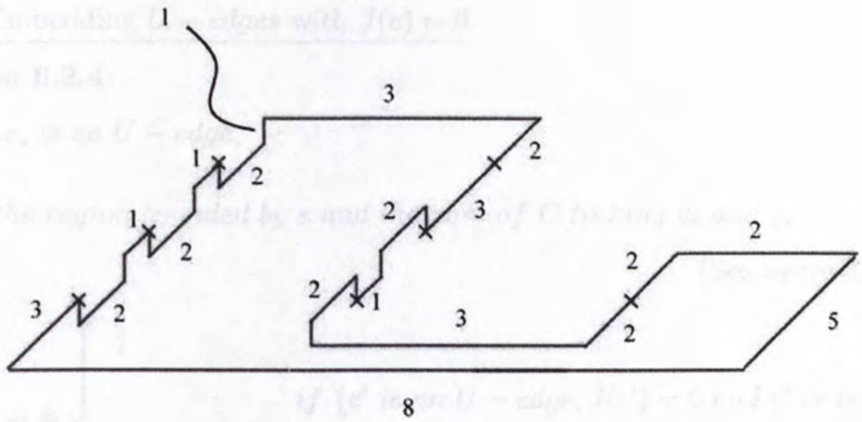


Figure 31d

For *type*(a), we will just use a straight line to link v_i and v_{i+1}. For *type*(b), we will first go down 1 step, moves 2 steps, go up 1 step and then a straight line. $Type(c)$ is similar with "going down" replaced by "going up". The purpose of this setting is to ensure all $U-edges$ embedded in $z \geq 0$ and all $B-edges$ embedded in $z \leq 0$ and to maintain the planar relation of the 4 edges incident with a vertex.

Definition 6.2.3:

e is called $v_r v_s$ $(r < s)$ if e is incident with v_r, v_s,

$v_r v_s$ is called jumps from column i to column j if
v_r is in column i and v_s is in column j,

$J(v_r v_s) \triangleq |i-j|$ if $v_r v_s$ jumps from column i to column j,
$J(e)$ is called the jump number of e

Step 3 : Embedding $U-edges$ with $J(e)=0$

Definition 6.2.4:

e as $v_r v_s$ is an $U-edge$,

$R_e \triangleq$ *the region bounded by e and the part of C linking* v_r *and* v_s

(See figure 31e),

$$Level(e) \triangleq \begin{cases} 1 & \\ & if\ \{e' \ is\ an\ U-edge, J(e')=0\ and\ e'\ is\ inside\ R_e\}=\emptyset \\ 1+max\ \{Level(e'):\ \ e'\ is\ an\ U-edge, J(e')=0\ and\ e'\ is\ inside\ R_e\} & \\ & if\ \{e' \ is\ an\ U-edge, J(e')=0\ and\ e'\ is\ inside\ R_e\}\neq\emptyset \end{cases}$$

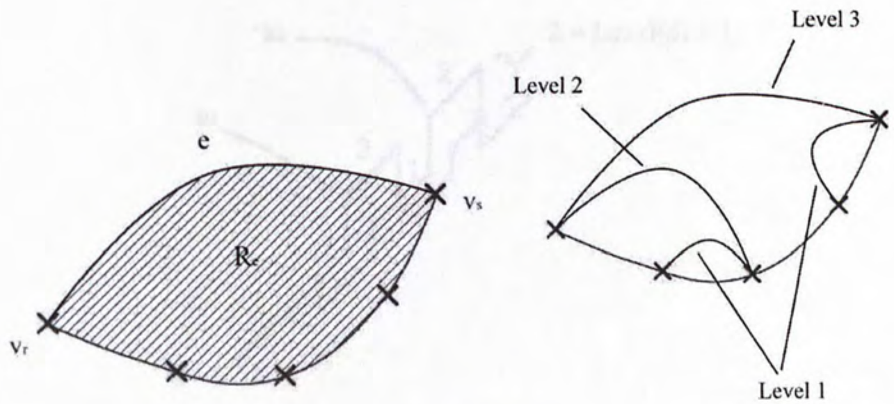


Figure 31e

Figure 31f illustrates the embedding of $U-edges$ with $J(e)=0$.

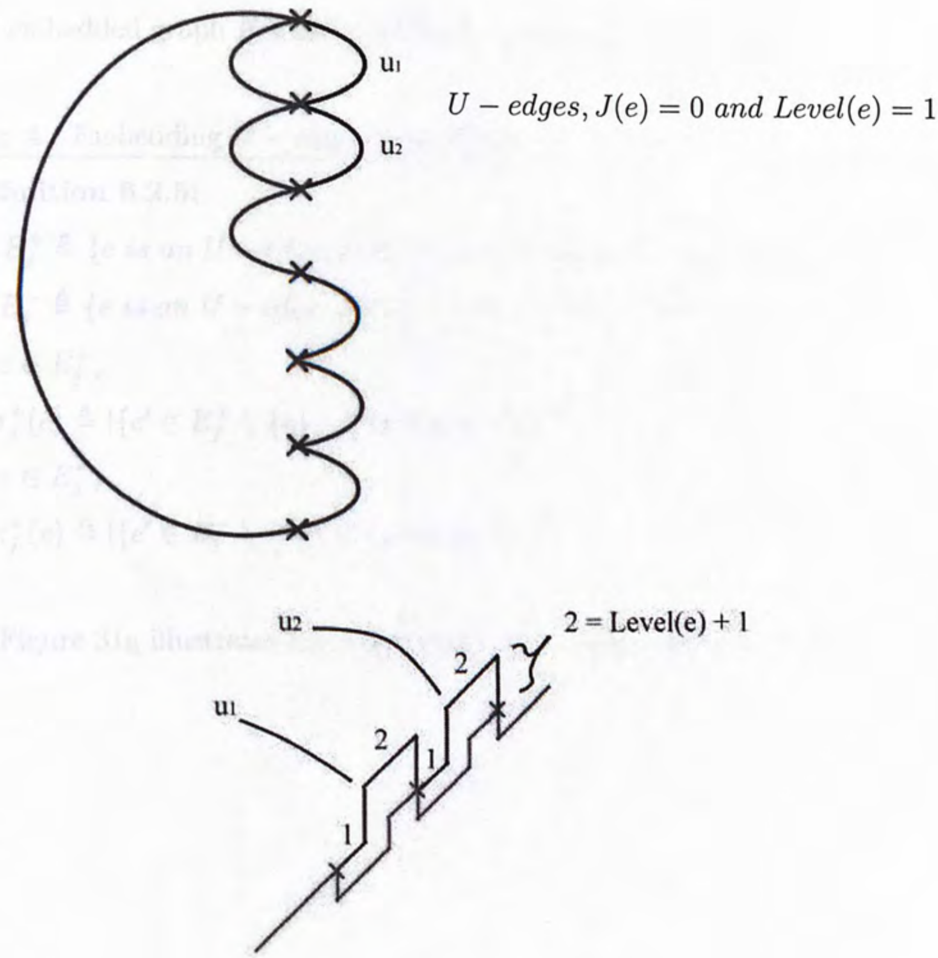


Figure 31f

Obviously, $Level(e)$ is used to keep track of the enclosing relation among them.

The edges are all embedded in $\mathbb{R}^2 \times [0, k+1]$ so far.

There is one very important remark to be noted here. By the planarity and non-intersecting property of the $U - edges$, any two edges e, e' will only have one of the following three relations : e enclosing e', e' enclosing e or neither of them enclosing the other. So, we may embed the $U - edges$ with $J(e) = 0$ first, $U - edges$ with $J(e) = 1$ and finally $U - edges$ with $J(e) \geq 2$. The former will always be enclosed by the latter or have no relation at all. By this construction, the embedded graph F will be evidently ambient isotopic to G.

Step 4 : Embedding $U-edges$ with $J(e)=1$

Definition 6.2.5:

$E_j^+ \triangleq \{e\ is\ an\ U-edge, J(e)\geq 1\ and\ e\ starts\ in\ column\ j\},$

$E_j^- \triangleq \{e\ is\ an\ U-edge, J(e)\geq 1\ and\ e\ ends\ in\ column\ j\},$

$e\in E_j^+,$

$t_j^+(e) \triangleq |\{e'\in E_j^+ \setminus \{e\} : e'\ is\ inside\ R_e\}|,$

$e\in E_j^-,$

$t_j^-(e) \triangleq |\{e'\in E_j^- \setminus \{e\} : e'\ is\ inside\ R_e\}|,$

Figure 31g illustrates the embedding of $U-edges$ with $J(e)=1$.

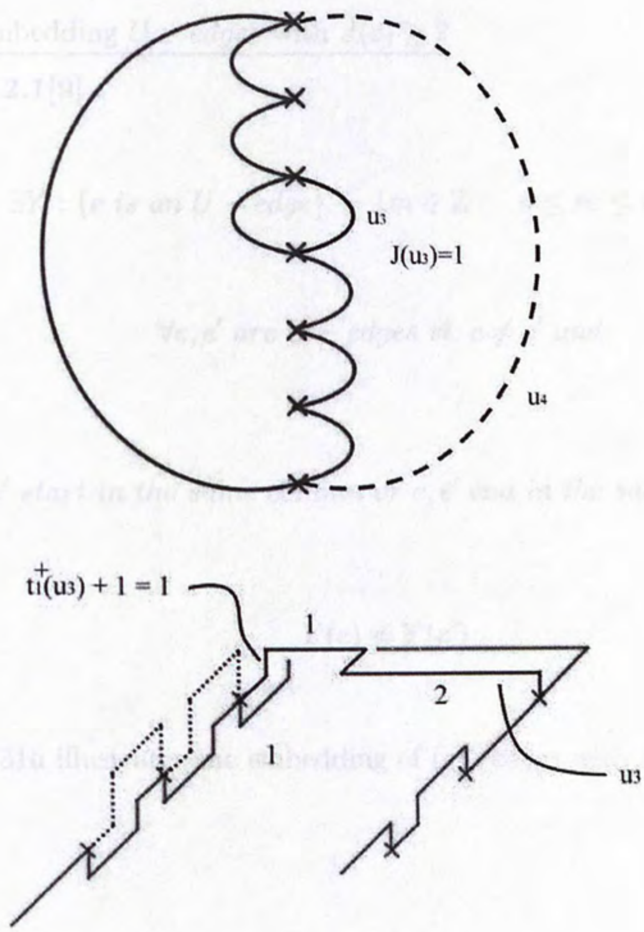


Figure 31g

$E_1^+ = \{u_3, u_4\}, E_2^+ = E_3^+ = \emptyset$

$E_1^- = \emptyset, E_2^- = \{u_3\}, E_3^- = \{u_4\}$

$t_1^+(u_3) = 0, t_1^+(u_4) = 1, t_2^-(u_3) = 0, t_3^-(u_4) = 0$

As before, $t_j^+(e)$ is used to keep track of the enclosing relation among the $U - edges$ with $J(e) \geq 1$.

The one right step is used to avoid collapsing with $U - edges$ with $J(e) = 0$.

The edges are all embedded in $\mathbb{R}^2 \times [0, 2k]$ so far.

Step 5 : Embedding $U - edges$ with $J(e) \geq 2$

Lemma 6.2.1[9] :

$$\exists Y : \{e \; is \; an \; U - edge\} \rightarrow \{m \in \mathbb{Z} : \quad 0 \leq m \leq 4k - 1\},$$

$$\forall e, e' \; are \; U - edges \; st. \; e \neq e' \; and$$

$$(e, e' \; start \; in \; the \; same \; column \; or \; e, e' \; end \; in \; the \; same \; column),$$

$$Y(e) \neq Y(e')$$

Figure 31h illustrates the embedding of $U - edges$ with $J(e) \geq 2$

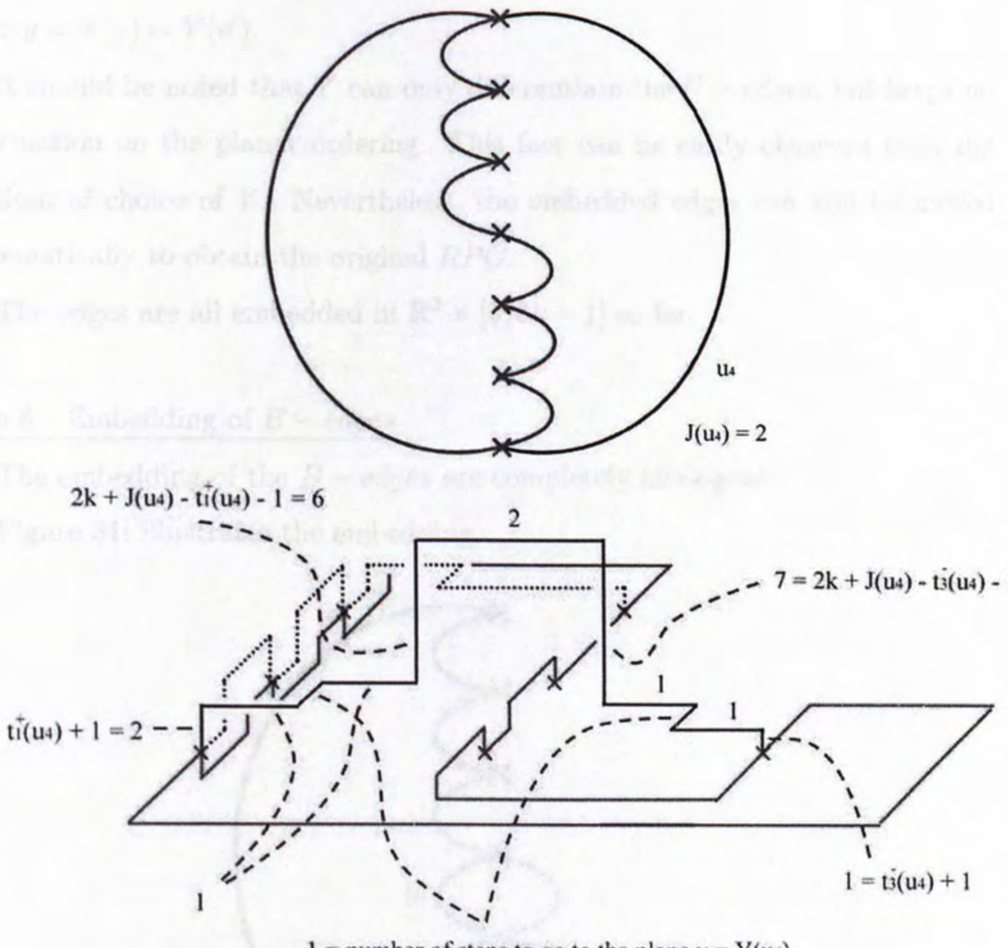


Figure 31h

$J(u_4) = 2$

In this example, we define Y as :

$Y : \{u_1, u_2, u_3, u_4\} \to \{m \in \mathbb{Z} : 0 \le m \le 11\}$ $st.$ $Y(u_i) \triangleq i$

The purpose of Y is to differentiate the $U-edges$ with $J(e) \ge 2$. By the function Y, all such $U-edges$ will be placed in different layers. Two edges will be embedded in the same layer if and only if they have no common starting column and ending column. No problem will be caused if neither of them contains the other. If e contains e', then $J(e) > J(e')$. This implies e will enclose e' in the layer $y = Y(e) = Y(e')$.

It should be noted that Y can only differentiate the $U-edges$, but keeps no information on the planar ordering. This fact can be easily observed from the freedom of choice of Y. Nevertheless, the embedded edges can still be moved systematically to obtain the original RPG.

The edges are all embedded in $\mathbb{R}^2 \times [0, 3k-1]$ so far.

Step 6 : Embedding of $B - edges$

The embedding of the $B - edges$ are completely analogous.

Figure 31i illustrates the embedding.

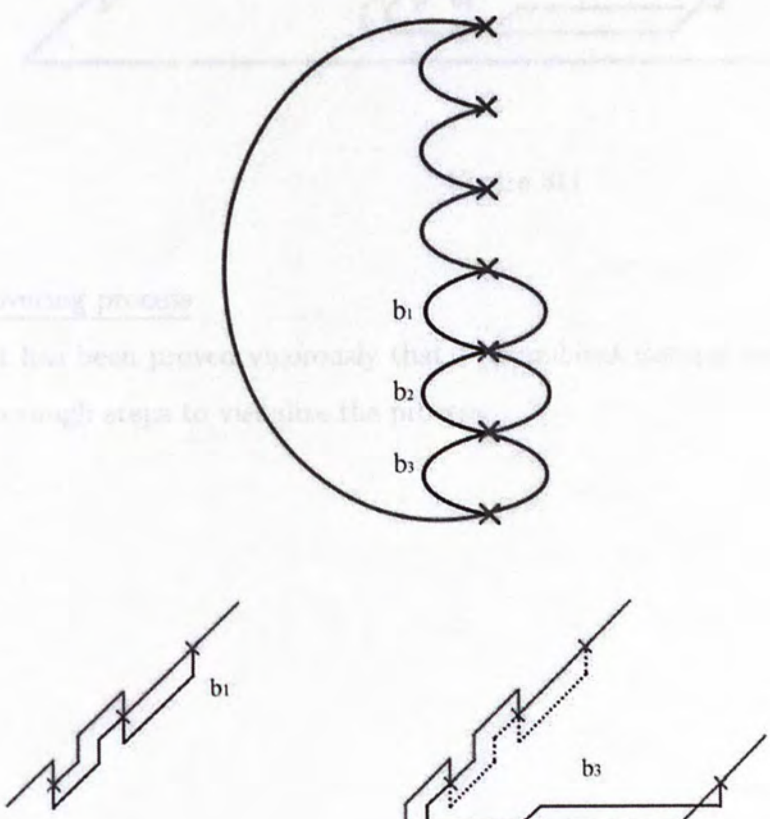


Figure 31i

$J(b_1) = J(b_2) = 0, J(b_3) = 1$

$Level(b_1) = Level(b_2) = 1$

$E_1^+ = E_3^+ = \emptyset, E_2^+ = \{b_3\}, t_2^+(b_2) = 0$

$E_1^- = E_2^- = \emptyset, E_3^- = \{b_3\}, t_3^-(b_3) = 0$

The final graph F will be :

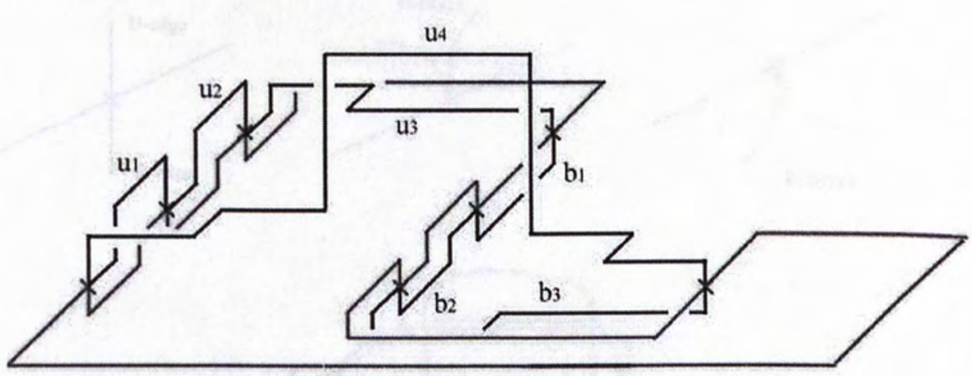


Figure 31j

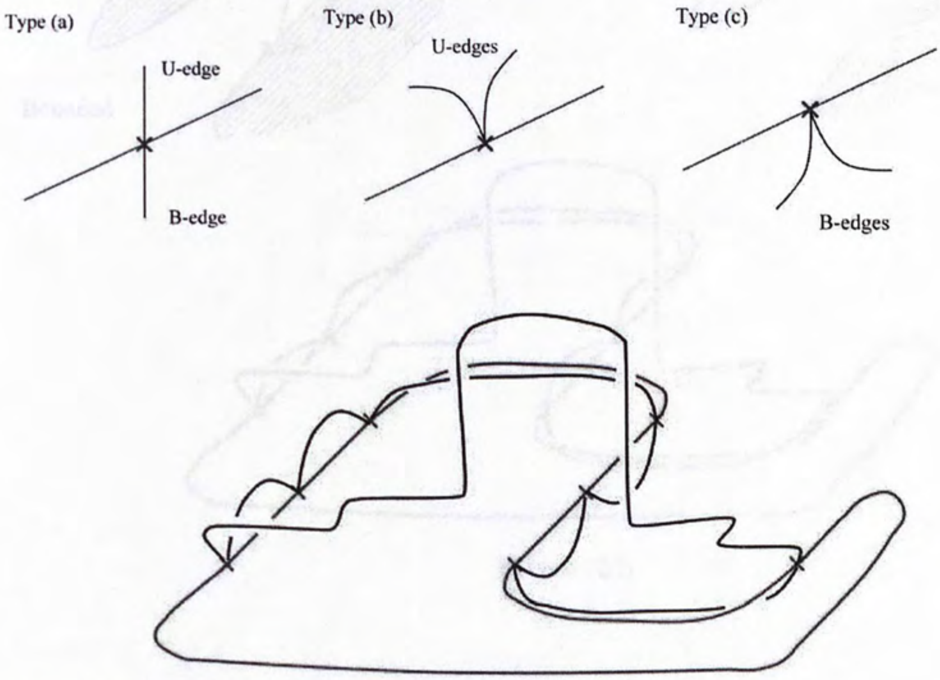
Recovering process

It has been proved vigorously that F is ambient isotopic to G. But here are some rough steps to visualize the process.

Step 1 : Smoothing F and straightening C

Figure 32a shows the rough result at v_i for $type(a), (b), (c)$.

Figure 32a



Step 2 : Moving edges with $J(e) = 0$

Firstly, e is rescaled to be of arbitrarily small height while keeping the level relation. Then, rotate e to the plane $z = 0$ depending on whether e is $U - edge$ or $B - edge$. (See figure 32b)

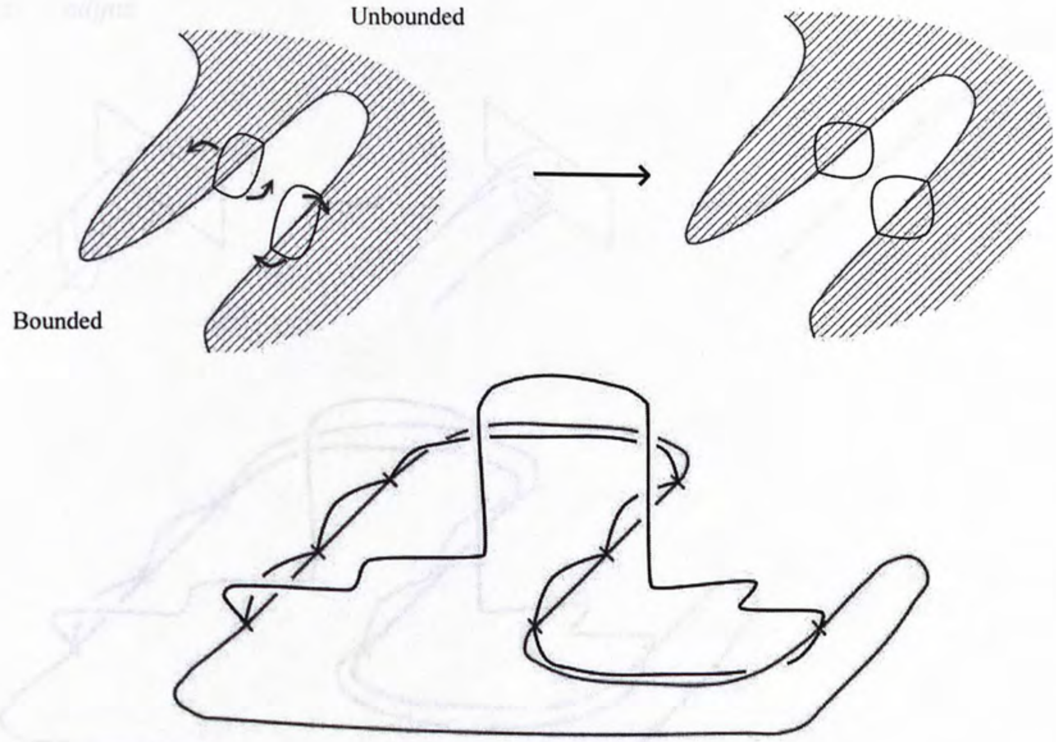


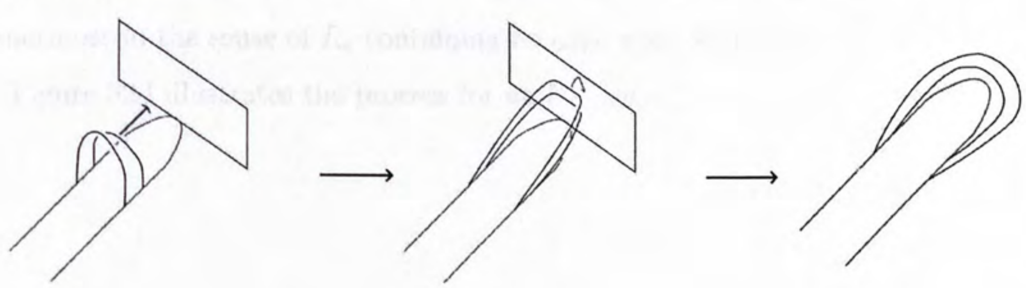
Figure 32b

Step 3 : Moving edges with $J(e) = 1$

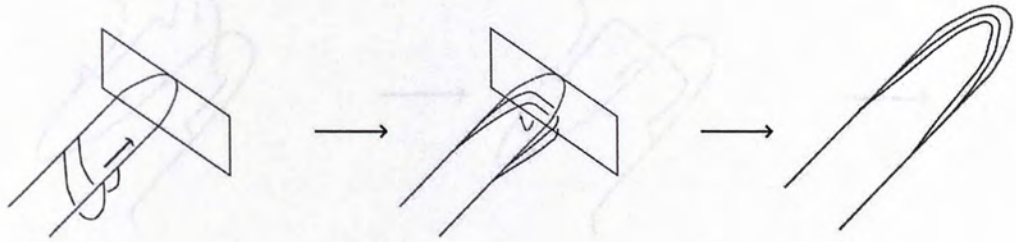
Figure 32c illustrates the transformation.

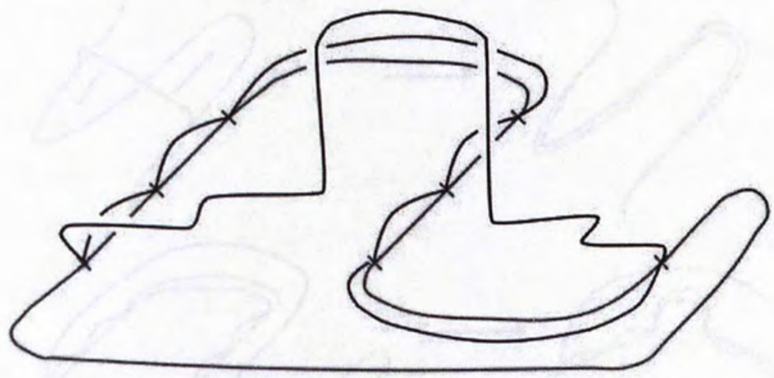
Figure 32c

$U - edges$



B − edges





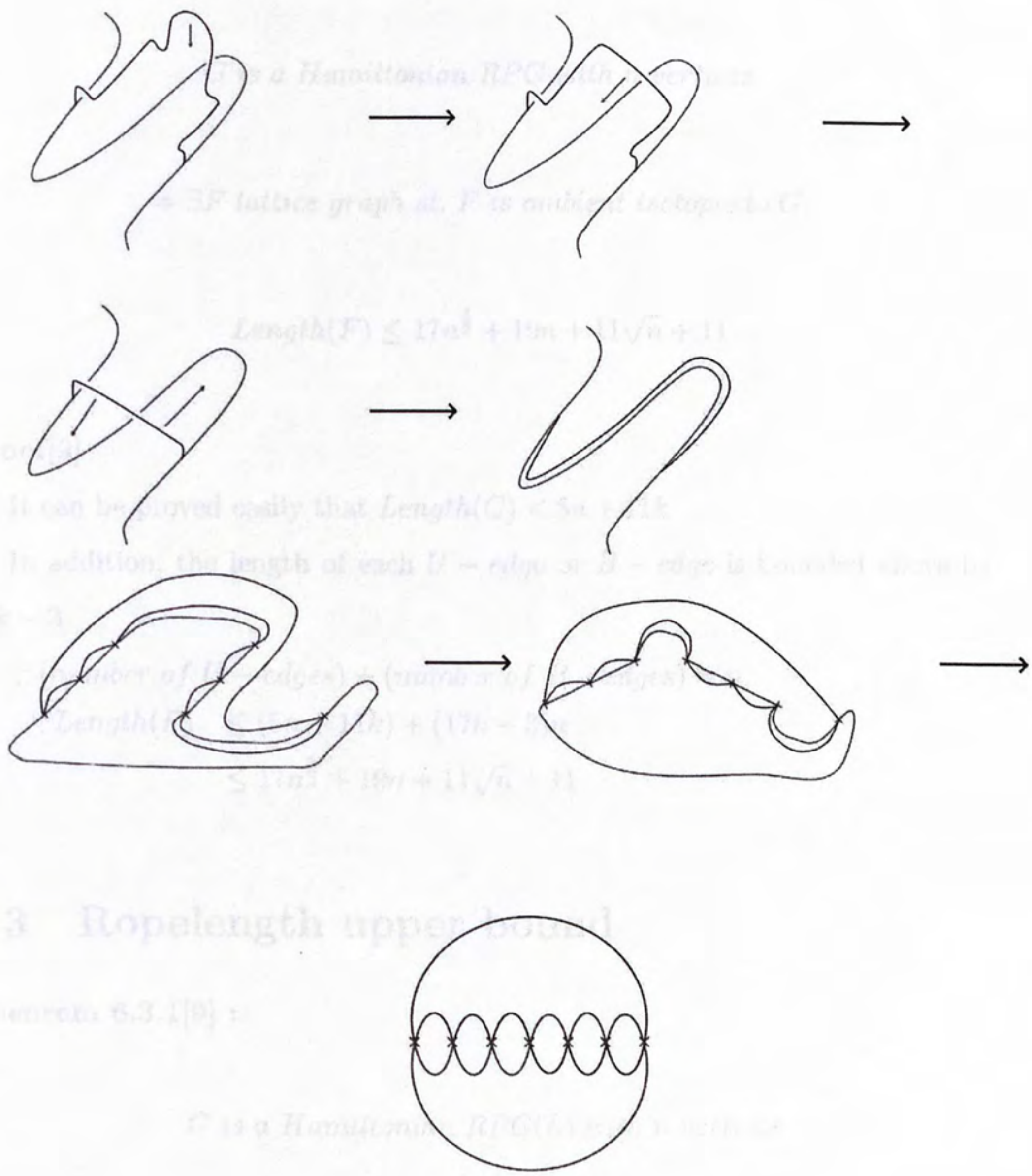
Since the $U - edges$ ($B - edges$) with $J(e) \geq 2$ are all above (below) the $U - edges$ ($B - edges$) with $J(e) = 1$ and edges with $J(e) = 0$ are already placed in $z = 0$ plane, this transformation will cause no problem.

Step 4 : Moving edges with $J(e) \geq 2$

We move the edges one by one starting with the innermost edge with $J(e) \geq 2$. (Innermost in the sense of R_e containing no edge with $J(e) \geq 2$)

Figure 32d illustrates the process for an $U - edge$.

Figure 32d



Because e as $v_r v_s$ is innermost, it is impossible to have e' as $v_{r'} v_{s'}$ st. $J(e') \geq 2$ and $r \leq r' < s' \leq s$. Accordingly, by careful examination with the definition of Y, this process can be done properly.

Theorem 6.2.1[9] :

$$G \text{ is a Hamiltonian RPG with } n \text{ vertices}$$

$$\Rightarrow \exists F \text{ lattice graph st. } F \text{ is ambient isotopic to } G,$$

$$Length(F) \leq 17n^{\frac{3}{2}} + 19n + 11\sqrt{n} + 11$$

Proof[9]

It can be proved easily that $Length(C) < 5n + 11k$

In addition, the length of each $U - edge$ or $B - edge$ is bounded above by $17k - 3$

$\because (number\ of\ U - edges) + (number\ of\ B - edges) = n,$

$$\begin{aligned} \therefore Length(F) &\leq (5n + 11k) + (17k - 3)n \\ &\leq 17n^{\frac{3}{2}} + 19n + 11\sqrt{n} + 11 \end{aligned}$$

6.3 Ropelength upper bound

Theorem 6.3.1[9] :

$$G \text{ is a Hamiltonian } RPG(L) \text{ with } n \text{ vertices}$$

$$\Rightarrow \exists F \text{ lattice graph st. } F \in [L],$$

$$L(F) \leq 17n^{\frac{3}{2}} + 21n + 11\sqrt{n} + 11$$

Proof[9]

As observed in the recovery process, we just have to tackle the crossing points of the lattice graph F' corresponding to G.

For each crossing point, we make one of two modifications depending on L as in figure 33 to obtain F. Then, obviously, F is ambient isotopic to L and

$L(F) \leq 17n^{\frac{3}{2}} + 21n + 11\sqrt{n} + 11.$

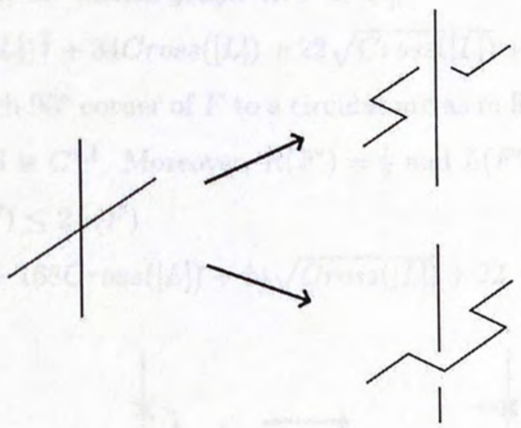


Figure 33

Theorem 6.3.2[9] :

$$\forall L, \exists F \ lattice\ graph\ st.\ F \in [L],$$

$$L(F) \leq 136Cross([L])^{\frac{3}{2}} + 84Cross([L]) + 22\sqrt{Cross([L])} + 11$$

Proof[9]

It follows directly from **Theorem 6.1.5** and **Theorem 6.3.1**.

Theorem 6.3.3[9] :

$$\forall L, \quad RL([L]) \leq 272Cross([L])^{\frac{3}{2}} + 168Cross([L]) + 44\sqrt{Cross([L])} + 22$$

Proof[9]

By **Theorem 6.3.2**, $\exists F$ *lattice graph st.* $F \in [L]$,

$L(F) \leq 136Cross([L])^{\frac{3}{2}} + 84Cross([L]) + 22\sqrt{Cross([L])} + 11$

By transforming each 90° corner of F to a circular arc as in figure 34 to obtain F', F' is still in $[L]$ and is $C^{1,1}$. Moreover, $R(F') = \frac{1}{2}$ and $L(F') \leq L(F)$.

$\therefore RL([L]) \leq RL(F') \leq 2L(F)$

$\leq 272Cross([L])^{\frac{3}{2}} + 168Cross([L]) + 44\sqrt{Cross([L])} + 22$

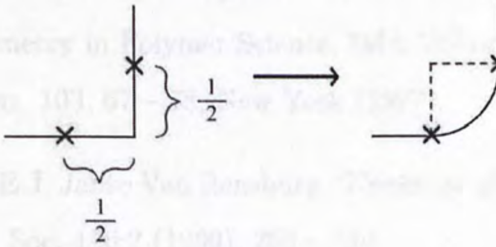


Figure 34

Remark :

Although the coefficients are rather large, the theorem gives an upper bound of $RL([L])$ with $O(Cross([L])^{\frac{3}{2}})$.

It was proved that $\exists a > 0, \exists \{K_n\}, \quad Cross([K_n]) \to \infty$ *and* $aCross([K_n]) \leq RL([K_n])$. So if $p \in \mathbb{R}^+$ *st.* $\exists M, \forall L, \quad \frac{RL([L])}{Cross([L])^p} < M$, p must be ≥ 1. Combining with the order achieved in this paper, we know that the minimum p satisfies $1 \leq p \leq \frac{3}{2}$.

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